Conformal embedding of a finite bordered Riemann surface

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1 Reduced Teichmüller space

Let R be a finite compact bordered Riemann surface of genus p with m boundary components and $\hat{R}$ be the doubled Riemann surface of R, which is compact Riemann surface of genus $2p+m-1$. Let i be an anticonformal mapping (involution) on $\hat{R}$ such that $i \circ i = identity$ and $R \cup i(R) = \hat{R}$. The reduced Teichmüller space of R_0 is

$$T(R_0) = \{(R, g); R \text{ is a finite compact bordered Riemann surface which}$$

$$\text{is mapped by a quasiconformal mapping g from } R_0 \text{ to } R\}/\sim,$$

where (R_1, g_1) is equivalent to (R_2, g_2) if there is a conformal mapping h from R_1 onto R_2 such that $g_2^{-1} \circ h \circ g_1$ is homotopic to the identity mapping.

We know $T(R_0)$ is a $6p-6+3m$ dimensional real analytic manifold.

For $R_i = (R_i, g_i) \in T(R_0)$, set

$$T(R_0; R_i) = \{R_j = (R_j, g_j) \in T(R_0); \text{there is a conformal mapping } f_j \text{ from}$$

$$R_i \text{ into a proper subdomain of } R_j \text{ with boundary}$$

$$\text{such that } g_j^{-1} \circ f_j \circ g_i \text{ is homotopic to the identity mapping}\}.$$

We request that f_j is conformal on ∂R_i and $f_j(\partial R_i)$ consists of analytic curves on R_j. Hence $R_j - f_j(R_i)$ has a positive measure. All Riemann surfaces by quasiconformal deformations on $R_j - f_j(R_i)$ cover a neighborhood V of R_j in $T(R_0)$. It is clear $V \subset T(R_0; R_i)$, hence $T(R_0; R_i)$ is open.

2 Optimal conformal embedding

For $(R_j, g_j) \in T(R_0; R_i)$, set

$$CE(R_i, R_j) = \{f; f \text{ is a conformal mapping from } R_i^\circ \text{ into } R_j^\circ \text{ such that}$$
$$g_j^{-1} \circ f \circ g_i \text{ is homotopic to the identity mapping}\},$$

where R_k° denotes the interior of R_k.

Let R'_j be a subdomain of R_j° such that the boundary is contained in R_j° and every component of $R_j^\circ - R'_j$ is doubly connected. For a R'_j, consider the following curve family;

$$Z(R_j^\circ, R'_j) = \{\gamma; \gamma \text{ consists of a family of rectifiable closed Jordan curves}$$
$$\text{each of which divides the boundary components of a component of}$$
$$R_j^\circ - R'_j \text{ from others and } \gamma \text{ divides all the components}\}.$$

Denote the extremal length of $Z(R_j^\circ, R'_j)$ by $\lambda(Z(R_j^\circ, R'_j))$, i.e.,

$$\lambda(Z(R_j^\circ, R'_j)) = sup_\rho\{\frac{1}{A(\rho)}; \rho \text{ is a Borel measurable conformal density}$$
$$\text{such that } \ inf_{\gamma \in Z(R_j^\circ, R'_j)}\{\int_\gamma \rho(z)|dz|\} \geq 1\},$$

where $A(\rho) = \int\int_{R_j} \rho^2(x+iy)dxdy$. If $(R_k, g_k) \sim (R'_k, g'_k)$, there is a conformal mapping h_k such that $g_k'^{-1} \circ h_k \circ g_k$ is homotopic to the identity mapping. Note that for $f \in CE(R_i, R_j)$,

$$\lambda(Z(R_j^\circ, f(R_i^\circ))) = \lambda(Z(R_j'^\circ, h_j \circ f \circ h_i^{-1}(R_i'^\circ))).$$

Put

$$B(R_i, R_j) = inf\{\lambda(Z(R_j^\circ, f(R_i^\circ))); f \in CE(R_i, R_j)\},$$

where $B(R_i, R_j) = \infty$ if $CE(R_i, R_j)$ is empty. We have

Theorem 1. *Suppose $B(R_i, R_j) < \infty$. There is an $f_{ij} \in CE(R_i, R_j)$ which satisfies $\lambda(Z(R_j^\circ, f_{ij}(R_i^\circ))) = B(R_i, R_j)$. The boundary of $f_{ij}(R_i^\circ)$ consists of trajectories of a quadratic holomorphic differential on R_j; hence the boundary is analytic.*

We call $f_{ij}(R_i)$ an optimal conformal embedding from R_i to R_j.

Let H be a harmonic function on $R_j - f_{ij}(R_i)$ such that H takes value one on the boundary of $f(R_j)$ and vanishes on the boundary of $f_{ij}(R_i)$. $(\frac{\partial}{\partial z}H)^2dz^2$ coincides with a quadratic holomorphic differential which is stated in the Theorem. The components of $R_j - f_{ij}(R_i)$ will be all annuli. For a t $(0 < t \leq 1)$ set $R_{jt} = f_{ij}(R_i) \cup \{z \in R_j - f_{ij}(R_i); H(z) \leq t\}$. Then $R_{jt} \in T(R_0; R_i)$ and R_{jt} , $R_{j1} = R_j$ are arcwise connected in $T(R_0; R_i)$. For the other $R_k \in T(R_0; R_i)$, similarly $R_{kt'}$ and R_k are arcwise connected in $T(R_0; R_i)$. For small t, we can choose t' so that $R_{kt'} \subset R_{jt}$. Since $R_{kt'}$ and R_{jt} become arcwise connected, R_k and R_j are arcwise connected in $T(R_0; R_i)$. It follows that

Proposition 2.

$T(R_0; R_i)$ is a $6p-6+3m$ dimensional real analytic submanifold of $T(R_0)$.

Suppose there exists $R_\infty (\neq R_i)$ which belongs to $\partial T(R_0; R_i) \subset T(R_0)$. We can take a sequence $R_n \in T(R_0; R_i)$ converging to R_∞. There are quasiconformal mappings h_n from R_n to R_∞ such that $g_\infty^{-1} \circ h_n \circ g_n$ are homotopic to the identity mapping and

$$\lim_{n\to\infty} esssup|\frac{h_{n\bar{z}}}{h_{nz}}| = 0.$$

For an optimal conformal embedding f_{in}, $h_n \circ f_{in}$ is a quasiconformal mapping from R_i to R_∞. A subsequence $\{h_k \circ f_{ik}\}$ may converge to a conformal mapping h_∞ from R_i° into R_∞. If $R_\infty - h_\infty(R_i)$ has a positive measure, then it is shown $R_\infty \in T(R_0; R_i)$. This is a contradiction. Therefore $h_\infty(\partial R_i)$ consists of boundary of R_∞ which may be added slits.

Ishida's example

Let

$$G = \hat{\mathbf{C}} - [-1, 0] \cup [1, 2] \cup [3, \infty],$$

$$G_\epsilon = \hat{\mathbf{C}} - [-1, 0] \cup [1, 2 + \epsilon] \cup [3 - \epsilon.\infty] \ (0 < \epsilon < \frac{1}{2}).$$

There is no subdomain G' of G such that G' is conformal to G_ϵ and $G - G'$ has a positive measure.

Suppose that there exists a subdomain G' satisfying the condition. Let

$$Z = \{\gamma; \gamma \text{ is a closed Jordan curve in } G$$

$$\text{which devide } [-1, 0] \text{ and } [1, 2] \cup [3, \infty]\},$$

$$Z_\epsilon = \{\gamma; \gamma \text{ is a closed Jordan curve in } G_\epsilon$$

$$\text{which devide } [-1,0] \text{ and } [1,2+\epsilon]\cup[3-\epsilon,\infty]\},$$

$$Z' = \{\gamma;\gamma \text{ is a closed Jordan curve in } G'$$

$$\text{which devide } [-1,0] \text{ and } [1,2]\cup[3,\infty]\}.$$

Take a harmonic function u on $\mathbf{C} - [-1,\hat{0}] \cup [1,\infty]$ whose boundary value takes 1 on $[-1,0]$ and 0 on $[1,\infty]$. We have

$$\lambda(Z) = \|du\|^2 = \lambda(Z_\epsilon) = \lambda(Z').$$

For an admissible density $\rho(x+iy) = \sqrt{u_x^2 + u_y^2}$,

$$\inf_{\gamma\in Z'} \int_\gamma \rho(x+iy)|d(x+iy)| \geq \inf_{\gamma\in Z} \int_\gamma \rho(x+iy)|d(x+iy)|,$$

$$\int\int_{G'} \rho(x+iy)^2 dxdy < \int\int_G \rho(x+iy)^2 dxdy.$$

Hence

$$\lambda(Z') \geq \frac{(\inf_{\gamma\in Z'} \int_\gamma \rho(x+iy)|d(x+iy)|)^2}{\int\int_{G'} \rho(x+iy)^2 dxdy}$$

$$> \frac{(\inf_{\gamma\in Z} \int_\gamma \rho(x+iy)|d(x+iy)|)^2}{\int\int_{G} \rho(x+iy)^2 dxdy} = \lambda(Z).$$

This is a contradiction.

3 Embeddability

Let Λ be the real Hilbert space of square integrable complex differentials whose inner product is given by

$$<\omega,\sigma>= Real\ part\ of \int\int_R \omega \bigwedge *\bar{\sigma} = \Re(\omega,\sigma),$$

where $*\sigma$ denotes the harmonic conjugate differential of σ and $\bar{\sigma}$ denotes the complex conjugate of σ. The following subspaces of Λ will be used

$$\Lambda_h = \{\lambda \in \Lambda : \lambda \text{ is a complex harmonic differential}\},$$

$$\Lambda_{eo} = \{\lambda \in \Lambda : \lambda \text{ is a closed differential which is orthogonal to } \Lambda_h,$$

$$\Gamma_h = \{\lambda \in \Lambda_h : \lambda \text{ is a real differential}\},$$

For a cycle C in a Riemann surface, let

$$Z(C) = \{\gamma; \gamma \text{ is a sum of rectifiable oriented curves}$$

$$\text{which is homologous to } C\}.$$

Denote the extremal length of $Z(C)$ by $\lambda(Z(C))$. Let $\sigma(C) = \sigma_R(C)$ be the period reproducing harmonic differential on R i.e.

$$\int_C \omega = (\omega, \sigma(C)) \text{ for } \omega \in \Gamma_h(R).$$

Then we know

Accola's Theorem

$$\lambda(Z(C)) = (\sigma(C), \sigma(C)) = \|\sigma(C)\|^2.$$

The following is clear by Theorem 1.

Lemma 3. *For every cycle C in R_0 and $R_k \in T(R_0; R_i)$*

$$\lambda(Z(g_k(C))) < \lambda(Z(g_i(C))).$$

Set

$$\tilde{T}(R_0; R_i) = \{R_j \in T(R_0); \lambda(Z(g_j(C))) < \lambda(Z(g_i(C))) \text{ for every cycle } C\}$$

By Lemma $\tilde{T}(R_0; R_i)$ contains $T(R_0; R_i)$. Can we expect $\tilde{T}(R_0; R_i) = T(R_0; R_i)$? The case of annulus is trivially valid. For the case of once hold torus, depending on Shiba's results, M. Masumoto gives the following.

Masumoto's Theorem

$$T(R_0; R_i) = \{(X_j, Y_j, Z_j); F(X_i - X_j, Y_i - Y_j, Z_i - Z_j) < 0,$$

$$F(X_i, Y_i, Z_i) < F(X_j, Y_j, Z_j)\},$$

where

$$F(X, Y, Z) = X^2 + Y^2 + Z^2 - 2XY - 2YZ - 2ZX,$$

for a canonical homology basis $\{A, B\}$ *of* R_0

$$X_k = \lambda(Z(g_k(A))),\ Y_k = \lambda(Z(g_k(B))),\ Z_k = \lambda(Z(g_k(A) - g_k(B))),$$

Consider the case of triply connected region $R_0(p = 0, m = 3)$. Let $\{B_1, B_2\}$ be a canonical homology basis of R_0. Set

$$X_k = \lambda(Z(g_k(B_1))),\ Y_k = \lambda(Z(g_k(B_2))),\ Z_k = \lambda(Z(g_k(B_1) + g_k(B_2))),$$

We use the same notation C in R_i and regard $C = q_1B_1 + q_2B_2$ a cycle in $\hat{R}_i$. For the period reproducing harmonic differential $\sigma(C) = \sigma_{\hat{R}_i}(C)$ on $\hat{R}_i$

$$\|\sigma(C)\|^2 = q_1^2\|\sigma(B_1)\|^2 + 2q_1q_2(\sigma(B_1), \sigma(B_2))\|^2 + q_2^2\|\sigma(B_2)\|^2.$$

By

$$2(\sigma(B_1), \sigma(B_2)) = \|\sigma(B_1) + \sigma(B_2)\|^2 - \|\sigma(B_1)\|^2 - \|\sigma(B_2)\|^2,$$

we get

$$0 \leq \|\sigma(C)\|^2 - \|\sigma(g_j \circ g_i^{-1}(C))\|^2$$
$$= 2((X_i - X_j)q_1^2 + (Z_i - Z_j - (X_i - X_j) - (Y_i - Y_j))q_1q_2 + (Y_i - Y_j)q_2^2).$$

It follows that $X_i - X_j > 0$ and

$$F(X_i - X_j, Y_i - Y_j, Z_i - Z_j) < 0.$$

Take a canonical homology basis $\{A_k, B_k\}_{k=1,\dots,p}$ and $\{B_k\}_{k=2p+1,\dots,2p+m-1}$ of R, where $B_{2p+\ell}$ is homologus to some boundary component. Set $A_{p+k} = -i(A_k)$ and $B_{p+k} = i(B_k)$. Further we can take $\{A_k\}_{k=2p+1,\dots,2p+m-1}$ in $\hat{R}$ such that $\{A_k, B_k\}_{k=1,\dots,2p+m-1}$ is a canonical homology basis of $\hat{R}$.

Denote

$$\alpha_{k,\ell}(R) = (\sigma_R(A_k), \sigma_R(A_\ell))_R,\ \tilde{\alpha}_{k,\ell}(R) = (\sigma_R(A_k), \sigma_R(B_\ell))_R = \beta_{\ell,k}(R),$$

$$\beta_{k,\ell}(R) = (\sigma_R(B_k), \sigma_R(A_\ell))_R,\ \tilde{\beta}_{k,\ell}(R) = (\sigma_R(B_k), \sigma_R(B_\ell))_R = \tilde{\beta}_{\ell,k}(R),$$

$$\tilde{\alpha}_{k,2p+\ell}(R) = (\sigma_R(A_k), \sigma_R(B_{2p+\ell}))_R,\ \beta_{2p+k,\ell}(R) = (\sigma_R(B_{2p+k}), \sigma_R(A_\ell))_R = \tilde{\alpha}_{\ell,2p+k}(R),$$

$$\tilde{\beta}_{2p+k,\ell}(R) = (\sigma_R(B_{2p+k}), \sigma_R(B_\ell))_R,\ \tilde{\beta}_{2p+k,2p+\ell}(R) = (\sigma_R(B_{2p+k}), \sigma_R(B_{2p+\ell}))_R,$$

and matrices

$$\mathbf{A}(R) = (\alpha_{k,\ell}(R)),\ \tilde{\mathbf{A}}(R) = (\tilde{\alpha}_{k,\ell}(R)),\ \mathbf{B}(R) = (\beta_{k,\ell}(R)),\ \tilde{\mathbf{B}}(R) = (\tilde{\beta}_{k,\ell}(R)).$$

$$\mathbf{C}(R) = (\beta_{2p+k,\ell}(R)),\ \tilde{\mathbf{C}}(R) = (\tilde{\beta}_{2p+k,\ell}(R)),\ \tilde{\mathbf{D}}(R) = (\tilde{\beta}_{2p+k,2p+\ell}(R)).$$

Set

$$\mathbf{S}(R) = \begin{pmatrix} \mathbf{A}(R) & {}^t\mathbf{B}(R) & {}^t\mathbf{C}(R) \\ \mathbf{B}(R) & \tilde{\mathbf{B}}(R) & {}^t\tilde{\mathbf{C}}(R) \\ \mathbf{C}(R) & \tilde{\mathbf{C}}(R) & \tilde{\mathbf{D}}(R) \end{pmatrix}.$$

For a cycle

$$C = \sum_{k=1}^{p}(t_k A_k + s_k B_k) + \sum_{k=1}^{m-1} s_{2p+k} B_{2p+k},$$

$$\lambda(Z(C)) = \|\sigma_R(C)\|^2 = \int_C \sigma_R(C) = \mathbf{x}\mathbf{S}(R)^t\mathbf{x},$$

where

$$\mathbf{x} = (t_1, ..., t_p, s_1, ..., s_p, s_{2p+1}, ..., s_{2p+m-1}),$$

It follows that $\mathbf{S}(R)$ is positive definite.

We have

Proposition 5. *If $R_j \in T(R_0; R_i)$, then $\mathbf{S}(R_i) - \mathbf{S}(R_j)$ is positive definite.*

Take a canonical homology basis $\{A_k, B_k\}_{k=1,...,p}$ and $\{B_k\}_{k=2p+1,...,2p+m-1}$ of R, where $B_{2p+\ell}$ is homologus to some boundary component. Set $A_{p+k} = -i(A_k)$ and $B_{p+k} = i(B_k)$. Further we can take $\{A_k\}_{k=2p+1,...,2p+m-1}$ in $\hat{R}$ such that $\{A_k, B_k\}_{k=1,...,2p+m-1}$ is a canonical homology basis of $\hat{R}$.

Let

$$\tilde{\psi}_k = \sigma_R(A_k) + i^*\sigma_R(A_k),\ \tilde{\psi}_{p+k} = \sigma_R(B_k) + i^*\sigma_R(B_k),\ \tilde{\psi}_{2p+k} = \sigma_R(B_{2p+k}) + i^*\sigma_R(B_{2p+k}).$$

We have

$$\int_{A_\ell} \tilde{\psi}_k = \begin{cases} \alpha_{k,\ell}, & k = 1, ..., p \\ \beta_{k-p,\ell} - i\delta_{k-p,\ell}, & k = p+1, ..., 2p \\ \beta_{k,\ell}, & k = 2p+1, ..., 2p+m-1 \end{cases},$$

$$\int_{B_\ell} \tilde{\psi}_k = \begin{cases} \tilde{\alpha}_{k,\ell} + i\delta_{k,\ell}, & k = 1, ..., p \\ \tilde{\beta}_{k-p,\ell}, & k = p+1, ..., 2p \\ \tilde{\beta}_{k,\ell}, & k = 2p+1, ..., 2p+m-1 \end{cases},$$

$$\int_{B_{2p+\ell}} \tilde{\psi}_k = \begin{cases} \tilde{\beta}_{2p+k,\ell}, & k = 1, ..., p \\ \tilde{\beta}_{p+k,\ell}, & k = p+1, ..., 2p \\ \tilde{\beta}_{k,2p+\ell}, & k = 2p+1, ..., 2p+m-1 \end{cases}.$$

Period reproducer belongs to ${}^*\Gamma_{ho}$ and can be symmetrically extended to the doubled Riemann surface. On the other hand, the element of Γ_{ho} can be anti-symmetrically extended to the doubled Riemann surface. Hence $\tilde{\psi}_k$ is extended to the doubled Riemann surface and denote it $\hat{\psi}_k$. Then

$$\int_{A_{2p+\ell}} \hat{\psi}_k = \begin{cases} 0, & k = 1, ..., p \\ 0, & k = p+1, ..., 2p \\ 2\delta_{k,2p+\ell}, & k = 2p+1, ..., 2p+m-1 \end{cases},$$

$$\int_{A_{p+\ell}} \hat{\psi}_k = \begin{cases} -\alpha_{k,\ell}, & k = 1, ..., p \\ -\beta_{k-p,\ell} - i\delta_{k-p,\ell}, & k = p+1, ..., 2p \\ -\beta_{k,\ell}, & k = 2p+1, ..., 2p+m-1 \end{cases},$$

$$\int_{B_{p+\ell}} \hat{\psi}_k = \begin{cases} \tilde{\alpha}_{k,\ell} - i\delta_{k,\ell}, & k = 1, ..., p \\ \tilde{\beta}_{k-p,\ell}, & k = p+1, ..., 2p \\ \tilde{\beta}_{k,\ell}, & k = 2p+1, ..., 2p+m-1 \end{cases}.$$

We know $\{\hat{\psi}_k\}$ is linearly independent over complex number field and they are a basis of holomorphic differentials on the doubled Riemann surface. It follows that by Torelli's theorem

Proposition 6. *If* $\mathbf{S}(R_i) = \mathbf{S}(R_j)$, *$R_i$ and R_j are conformally equivalent.*

Since Riemann's period matrix is represented by inner products of some period reproducing harmonic differentials, we have

Corollary 7.

$R_i = R_k$ in $T(R_0)$ if

$$\lambda(Z(g_k(C))) = \lambda(Z(g_i(C))) \text{ for every cycle } C \text{ in } R_0.$$

For $(R, g) \in T(R_0)$, let

$$I(R) = \{\tilde{R} : \tilde{R} \text{ is a compact Riemann surface of ginus } p \text{ in which } R \text{ is embeded}\}$$

and $\tilde{i}$ be the embedding conformal mapping from R into $\tilde{R}$. For homology basis $\{\tilde{i} \circ g(A_k), \tilde{i} \circ g(B_k)\}$ in $\tilde{R}$, take the Riemann's Period Matrix $\mathbf{T}(\tilde{R})$. Let $\mathbf{P}(R) = \{\mathbf{T}(\tilde{R}) : \tilde{R} \in I(R)\}$. If $R_j \in T(R_0, R_i)$, then $\mathbf{P}(R_j) \subset \mathbf{P}(R_i)$. Does the converse valid?

4 Punctured extensions

A curve family $\{\gamma_k\}_{k=1}^N$ in R is admissible if every γ_k is disjoint and not freely homotopic to the other γ_ℓ. Assume that every boundary component C_j is contained in the admissible curve family and let $\gamma_j = C_j, j = 1, ..., m$. There is a quadratic differential φ on R_i with closed trajectory such that closed trajectories freely homotopic to $g_i(\gamma_k)$ constructs ring domain $\{A_k\}$ and their ring domains divide R_i disjointedly and almost. Let $a_k = a_k(\varphi) = inf\{\int_\gamma |\varphi|^{1/2}; \gamma$ is freely homotopic to $g_i(\gamma_k)\}$. We can take charts $\{z_j\}_{j=1}^m$ of $\{A_j\}_{j=1}^m$ such that $\varphi_j = -(\dfrac{a_j dz_j}{2\pi z_j})^2$ and attach punctured disks $\{z_j; 0 < |z_j| < c_j\}$ to R_i and get a punctured Riemann surface $\tilde{R}_i = R_i(\varphi)$, which we call it punctured extension by quadratic differential with closed trajectory. The φ is regarded a quadratic differential on $\tilde{R}_i$ with closed trajectory and let $\{A'_j\}$ be the punctured disk in $\tilde{R}_i$ which consists of the closed trajectories freely homotopic to $g_i(\gamma_k)$.

Let $m'_j(\tilde{R}_i; \varphi)$, $m''_j(\tilde{R}_i; \varphi)$ be the reduced moduli for punctured disks A'_j, $\{A'_j - A_j\}$ with respect to some fixed parameter at the puncture and $m_k = m_k(\tilde{R}_i; \varphi)$ be the module of ring domain A_k. We denote $PE(R_i)$ such all punctured extensions of R_i and admissible curve family contained all boundary component. by quadratic differential with closed trajectory. If $R_\ell \in T(R_0; R_i)$, by theorem 1, there exists a punctured extension in $PE(R_i) \cap PE(R_\ell) \neq \emptyset$ and a quadratic differential φ with closed trajectory such that every boundary component of R_i and R_ℓ is the closed trajectory and

$$m''_j(\tilde{R}_i; \varphi) > m''_j(\tilde{R}_\ell; \varphi), \ j = 1, ..., m.$$

Proposition 8. *Let $R_i(\varphi_i) = R_\ell(\varphi_\ell) \in PE(R_i) \cap PE(R_\ell)$ under the same admissible curve family. If*

$$\sum a_j(\varphi_i)^2 m'_j(\tilde{R}_\ell; \varphi_\ell) + \sum a_k(\varphi_i)^2 m_k(\tilde{R}_\ell; \varphi_\ell)$$

$$= \sum a_j(\varphi_i)^2 m'_j(\tilde{R}_i; \varphi_i) + \sum a_k(\varphi_i)^2 m_k(\tilde{R}_i; \varphi_i)$$

and

$$m''_j(\tilde{R}_i; \varphi_i) > m''_j(\tilde{R}_\ell; \varphi_\ell), \ j = 1, ..., m,$$

then $\varphi_i = \varphi_\ell$ and $R_\ell \in T(R_0; R_i)$.

5 Abelian Teichmüller disk

Let $\{A_k, B_k\}$ be a canonical homology basis of modulo dividing cycles , Γ_x be a subspace of Γ_h and $*\Gamma_x^\perp$ be the space of harmonic conjugate differentials which are orthogonal to every differential of Γ_x. Assume that $\Gamma_x = *\Gamma_x^\perp$ and every differential in Γ_x has vanishing A period and is semiexact. Set $\Lambda_x = \Gamma_x + i * \Gamma_x^\perp$ and call Λ_x a behavior space. A holomorphic differential ψ is said to have Λ_x-behavior if there exists $\omega \in \Lambda_x$ and $\omega_0 \in \Lambda_{eo}$ such that $\psi = \omega + \omega_0$ outside of a compact set. Let ϕ be a square integrable holomorphic differential and

$$AT(\phi) = \{t\frac{\bar{\phi}}{\phi}; |t| < 1\},$$

which is regarded as a submanifold of Teichmüller space and called an Abelian Teichmüller disk. Let g_t be a quasiconformal mapping from R_0 to R_t whose Bertrami differential is $t\frac{\bar{\phi}}{\phi}$.

Let $\Gamma_x(R_t)$ be the orthogonal projection to $\Gamma_h(R_t)$ of the pull back $\Gamma_x(R_0) \circ g_t^{-1}$. Then $\Lambda_x(R_t) \circ g_t \subseteq \Lambda_x(R_0) + \Lambda_{eo}(R_0)$ and there exists uniquely a holomorphic differential ψ_i^t with Λ_x-behavior such that

$$\int_{g_t(A_j)} \psi_i^t = \delta_{ij} \quad (\delta_{k,\ell} \text{ is Kroneker's delta}).$$

Set

$$\tau_{ij}(t;\phi) = \int_{g_t(B_j)} \psi_i^t.$$

Theorem 9. (cf. **Kra** [K])

Let ϕ and ϕ' be non zero holomorphic differential with Λ_x-behavior. Assume that for complex numbers t and s in the open unit disk,

$$\tau_{ij}(t;\phi) = \tau_{ij}(s;\phi') \textit{ for every } i \textit{ and } j.$$

Then

$$t\frac{\bar{\phi}}{\phi} = s\frac{\bar{\phi}'}{\phi'}$$

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