BEHAVIOR OF BERGMAN METRIC UNDER QUASICONFORMAL DEFORMATIONS

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1 Quasiconformal-holomorphic movement

We are concerned with the change of function theoretic quantities on Riemann surfaces induced by quasiconformal deformations depending on a complex parameter. As a quasiconformal deformation we consider Riemann surfaces with conformal structures decided by certain Beltrami differentials depending holomorphically on a complex parameter. Let R be an open Riemann surface and M(R) be the set of Beltrami differentials:

$$\{\mu = \mu(z)\frac{d\bar{z}}{dz}; \mu \text{ is measurable and } \|\mu\|_\infty = esssup_R|\mu(z)| < 1\}$$

From $\mu \in M(R)$ we get another Riemann surface R_μ with the Riemannian metric $ds = \lambda(z)|dz + \mu(z)d\bar{z}|$. We consider Beltrami differentials $\mu_t = \mu(z,t)d\bar{z}/dz \in M(R)$ with a complex parameter t varying in a domain about zero. We assume that the following condition (H) is satisfied[4 :

(i) $\mu(z,t)$ is measurable and $\mu(z,0) = 0$,

(ii) For every t, there exist positive numbers ϵ_t, M_t such that

$$|\epsilon| < \epsilon_t \Longrightarrow \|\mu_{t+\epsilon} - \mu_t\|_\infty < |\epsilon| M_t,$$

(iii) For almost all z, $\mu(z,t)$ is holomorphic with respect to t.

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We say Quasiconformal-holomorphic movement for such a family $\{R_t\}$ which satisfies the condition (H).

Let f_t be the quasiconformal mapping from R to $R_\mu(= R_t)$ whose Beltrami coefficient is $\mu(z,t) = (f_t)_{\bar{z}}/(f_t)_z$.

2 Behavior spaces

Let Λ be the real Hilbert space of square integrable complex differentials whose inner product is given by

$$< \omega, \sigma >= \mathit{Real\ part\ of} \iint_R \omega \bigwedge *\bar{\sigma} = \Re(\omega, \sigma),$$

where $*\sigma$ denotes the harmonic conjugate differential of σ and $\bar{\sigma}$ denotes the complex conjugate of σ. The following subspaces of Λ will be used:

$$\Lambda_h = \{\lambda \in \Lambda : \lambda \text{ is a complex harmonic differential}\},$$

$$\Lambda_{eo} = \{\lambda \in \Lambda : \lambda \text{ is a closed differential which is orthogonal to } \Lambda_h\},$$

$$\Gamma_h = \{\lambda \in \Lambda_h : \lambda \text{ is a real differential}\}.$$

Let Γ_x be a subspace of Γ_h and $*\Gamma_x^\perp$ be the space of harmonic conjugate differentials which are orthogonal to every differential of Γ_x. Set $\Lambda_x = \Gamma_x + i * \Gamma_x^\perp$ and call Λ_x a behavior space. Here we assume that $\Lambda_x(R_t) \circ h_t \subseteq \Lambda_x(R_0) + \Lambda_{eo}(R_0)$.

For example, $i\Gamma_h$ and Γ_h are behavior spaces and satisfy the condition.

3 Variation of meromorphic differentials with behavior

We have some variational formulas of specific kind of meromorphic differentials[4].

Theorem 1. *Let ϕ^t be meromorphic differentials such that $\phi^t \circ h_t - \phi^0 \in \Lambda_x + \Lambda_{eo}$. Assume that the support of Beltrami coefficient μ_t of h_t does not meet an open set V including poles of ϕ^0. Then for $t = u + iv$ there exist differentials $\phi_u^t, \phi_v^t \in \Lambda_x(R_t) + \Lambda_{eo}(R_t)$ such that*

$$\lim_{\tilde{u} \to 0} \| \frac{\phi^{t+\tilde{u}} \circ h_{t+\tilde{u}} \circ h_t^{-1} - \phi^t}{\tilde{u}} - \phi_u^t \| = 0,$$

$$\lim_{\tilde{v} \to 0} \| \frac{\phi^{t+i\tilde{v}} \circ h_{t+i\tilde{v}} \circ h_t^{-1} - \phi^t}{\tilde{v}} - \phi_v^t \| = 0,$$

where $\tilde{u}$ and $\tilde{v}$ are real. Further $\phi_u^t + i\phi_v^t = i * (\phi_u^t + i\phi_v^t)$ is a holomorphic differential.

Set
$$\frac{\partial \phi^t}{\partial t} = \frac{1}{2}(\phi_u^t - i\phi_v^t), \ \frac{\partial \phi^t}{\partial \bar{t}} = \frac{1}{2}(\phi_u^t + i\phi_v^t).$$

4 Canonical meromorphic differentials with behavior

Suppose that every support of μ_t does not meet a parametric disk $V = \{z : |z| < 1\}$ about p. We can regard z as a local variable of $V_t = h_t(V)$ in R_t. There exist the following canonical meromorphic differentials $\varphi_n^t, \ \psi_n^t$ on R_t with behavior:

(i) $\varphi_n^t - \frac{dz}{z^{n+1}}$ and $\psi_n^t - \frac{dz}{z^{n+1}}$ are holomorphic on V_t ,

(ii) φ_n^t coincides with an element of $i\Gamma_h + \Lambda_{eo}$ on $R_t - V_t$ and ψ_n^t coincides with an element of $\Gamma_h + \Lambda_{eo}$ on $R_t - V_t$.

Let χ be a C^∞-real function such that $\chi = 1$ on $\{z : |z| < \frac{1}{2}\}$ and $\chi = 0$ on $R_t - V_t$. Take the following orthogonal decomposition:

$$\frac{1}{2}(-\frac{1}{n}d\frac{\chi}{z^n} + \frac{i}{n} * d\frac{\chi}{z^n}) = \omega_1 + \omega_2, \ \omega_1 \in i\Gamma_h + \Lambda_{eo}, \ \omega_2 \in \Gamma_h + *\Lambda_{eo}$$

and set $\sigma = -\frac{1}{2n}d\frac{\chi}{z^n} - \omega_1 = -\frac{i}{2n} * d\frac{\chi}{z^n} + \omega_2$. Then σ is harmonic except for $h_t(p)$. Hence $\sigma + i * \sigma = \frac{1}{n}d\frac{-\chi}{z^n} - \omega_1 + i * \omega_2$ is a meromorphic differential with pole of order $n+1$ at only $h_t(p)$ and coincides with $i\Gamma_h + \Lambda_{eo}$ on $R_t - V_t$.

Therefore $\sigma + i * \sigma$ satisfies the condition of φ_n^t. Similarly, by the orthogonal decomposition:

$$\frac{1}{2}(-\frac{1}{n}d\frac{\chi}{z^n} + \frac{i}{n} * d\frac{\chi}{z^n}) = \omega_1' + \omega_2',\ \omega_1' \in \Gamma_h + \Lambda_{eo},\ \omega_2' \in i\Gamma_h + *\Lambda_{eo},$$

ψ_n^t is obtained.

Hereafter, for simplicity, we write $\varphi_1^t = \varphi^t$, $\psi_1^t = \psi^t$.

5 Bergman kernel

Let $K_t = \hat{K}_t dz$ be a Bergman kernel for a point $h_t(p)$ on R_t such that

$$(\omega, K_t) = \hat{\omega}(0)$$

for any square integrable holomorphic differential $\omega = \hat{\omega}dz$.

Theorem 2.[(5]

$$K_t = \frac{1}{4\pi}(\varphi^t - \psi^t).$$

For any holomorphic differential $\omega \in \Lambda$, let

$$(\omega, \varphi^t)_V = \lim_{\epsilon \to 0} \iint_{V - V_\epsilon} \omega \wedge *\overline{\varphi^t} = i \int_{\partial V} w\overline{\varphi^t},$$

where $V_\epsilon = \{z; |z| < \epsilon\}$, $\omega = dw$ on V. The real part of φ^t coincides with an element of Λ_{eo} on $R_t - V_t$ and put it σ_1. We have

$$(\omega, \sigma_1 + i * \sigma_1)_V = (*\omega, *\sigma_1)_V - i(\omega, *\sigma_1)_V$$

$$= 2i \iint_V dw \wedge \sigma_1 = 2i \int_{\partial V} w \Re\varphi^t,$$

$$(\omega, \varphi^t) = (\omega, \sigma_1 + i * \sigma_1) - (\omega, \sigma_1 + i * \sigma_1)_V + (\omega, \varphi^t)_V$$

$$= -2i \int_{\partial V} w \Re\varphi^t + i \int_{\partial V} w\overline{\varphi^t} = -i \int_{\partial V} w\varphi^t = 2\pi \frac{dw}{dz}(0) = 2\pi\hat{\omega}(0).$$

Similarly the imaginary part of ψ^t coincides with an element of Λ_{eo} on $R_t - V_t$, and put it τ_1. We have

$$(\omega, \psi^t) = (\omega, - * \tau_1 + i\tau_1) - (\omega, - * \tau_1 + i\tau_1)_V + (\omega, \psi^t)_V$$

$$= 2(\omega, *\tau_1)_V + (\omega, \psi^t)_V = -2\int_{\partial V} w\Im\psi^t + i\int_{\partial V} w\overline{\psi^t}$$

$$= i\int_{\partial V} w\varphi^t = -2\pi\frac{dw}{dz}(0) = -2\pi\hat{\omega}(0).$$

Therefore square integrable holomorphic differential $(\varphi^t - \psi^t)/4\pi$ is the Bergman kernel. Set $L_t = \varphi^t + \psi^t$. The following variational formulas hold(4.

Theorem 3.

$$\frac{\partial \hat{K}_t(p)}{\partial t} = (K_t, \frac{\partial K_t}{\partial \bar{t}}),$$

$$\frac{\partial^2 \hat{K}_t(p)}{\partial \bar{t}\partial t} = (\frac{\partial K_t}{\partial \bar{t}}, \frac{\partial K_t}{\partial \bar{t}}) + (\frac{\partial L_t}{\partial \bar{t}}, \frac{\partial L_t}{\partial \bar{t}}),$$

$$\frac{\partial^2 \log \hat{K}_t(p)}{\partial \bar{t}\partial t} = \frac{1}{\hat{K}_t(p)}\{(\frac{\partial K_t}{\partial \bar{t}}, \frac{\partial K_t}{\partial \bar{t}}) + (\frac{\partial L_t}{\partial \bar{t}}, \frac{\partial L_t}{\partial \bar{t}})\}$$

$$-\frac{1}{\hat{K}_t(p)^2}|(\frac{\partial K_t}{\partial \bar{t}}, K_t)|^2 \geq 0.$$

If $\log \hat{K}_t(p)$ is harmonic,

$$\frac{\partial L_t}{\partial \bar{t}} = 0 \text{ and } \frac{\partial K_t}{\partial \bar{t}} = (a + ib)K_t,$$

where a and b are real. Hence

$$\frac{\partial \psi^t}{\partial \bar{t}} = -\frac{\partial \varphi^t}{\partial \bar{t}} \text{ and } \frac{\partial K_t}{\partial \bar{t}} = \frac{1}{2\pi}\frac{\partial \varphi^t}{\partial \bar{t}}.$$

For $t = u + iv$ (u, v real),

$$\frac{\partial \varphi^t}{\partial u} - a\varphi^t + ib\psi^t = a\psi^t - ib\varphi^t - i\frac{\partial \varphi^t}{\partial v}.$$

Suppose R_0 has a boundary part which consist of an analytic curve. The left side is pure imaginary along the boundary part and the right side is pure real along the boundary part. Further suppose that there is a curve from the boundary point to p, which does not meet the support of μ_t. Then the both sides are holomorphic on the curve and vanish there. Hence $a = b = 0$. It holds

$$\frac{\partial \varphi^t}{\partial u} = \frac{\partial \varphi^t}{\partial v} = 0 \text{ outside of the support of } \mu_t.$$

Further

$$\frac{\partial \varphi^t}{\partial \bar{t}} = \frac{\partial \psi^t}{\partial \bar{t}} = \frac{\partial K_t}{\partial \bar{t}} = 0 \text{ on } R_t.$$

Hence φ^t, ψ^t *and* K_t are holomorphic with respect to t. The meromorphic function $w = \frac{\varphi^t}{\psi^t}$ on R_t is pure imaginary along the boundary. Thus R_t is represented by w as a covering surface with slits over the imaginary axis. Since $w = w(z,t)$ is holomorphic with respect to t for a fixed z, these slits are static and only branch points may move as t varies. The w has a following development at p:

$$w = \frac{\varphi^t}{\psi^t} = 1 + \sum_{n=1} c_n z^n.$$

Set

$$\hat{\varphi}^t = \frac{1}{z^2} + \sum_{n=0} a_n z^n, \quad \hat{\psi}^t = \frac{1}{z^2} + \sum_{n=0} b_n z^n,$$

then

$$\frac{1}{z^2} + \sum_{n=0} a_n z^n = (\frac{1}{z^2} + \sum_{n=0} b_n z^n)(1 + \sum_{n=1} c_n z^n)$$
$$= \frac{1}{z^2} + \frac{c_1}{z} + (b_0 + c_2) + (b_1 + c_3)z + \ldots \quad .$$

Hence $c_1 = 0,\ b_0 + c_2 = a_0$. It follows that

$$c_2 = a_0 - b_0 = \hat{K}_t(p) = (K_t, K_t) > 0.$$

Therefore p in the covering surface is a branch point of order 1. When R_t is a finite compact bordered Riemann surface, φ^t and ψ^t are extended to its doubled Riemann surface. We can regard the doubled compact Riemann surface as a covering surface by w. The point p lies on $w = 1$ and the image point of p by the involution of doubled surface lies on $w = -1$.

Now, on this covering surface, we have

$$\varphi^t = \tilde{\Phi}(t,w)dw = \tilde{\Phi}(t,w)(w_z dz + w_{\bar{z}} d\bar{z}),$$

$$0 = \frac{\partial \varphi^t}{\partial \bar{t}}$$

$$= (\frac{\partial \tilde{\Phi}(t,w)}{\partial \bar{t}} + \frac{\partial \tilde{\Phi}(t,w)}{\partial w}\frac{\partial w}{\partial \bar{t}})(w_z dz + w_{\bar{z}} d\bar{z}) + \tilde{\Phi}(t,w)(\frac{\partial w_z}{\partial \bar{t}} dz + \frac{\partial w_{\bar{z}}}{\partial \bar{t}} d\bar{z}).$$

Since

$$0 = \frac{\partial w}{\partial \bar{t}} = \frac{\partial w_z}{\partial \bar{t}} = \frac{\partial w_{\bar{z}}}{\partial \bar{t}},$$

it follows

$$0 = \frac{\partial \tilde{\Phi}(t, w)}{\partial \bar{t}}(w_z dz + w_{\bar{z}} d\bar{z}),$$

i.e.

$$\frac{\partial \tilde{\Phi}(t, w)}{\partial \bar{t}} = 0.$$

Thus, by Hartogs theorem, $\tilde{\Phi}(t, w)$ is holomorphic with respect to t and w. Since φ_t is pure imaginary along the boundary, $\tilde{\Phi}(t, w)$ is constant on the fixed boundary point on w with respect to t. Hence it is constant on every surface point on w with respect to t. Now suppose that a branch point move as t varies. Then $\tilde{\Phi}(t, w)$ has the same Taylor development on the both sheets which contains the branch point. When the Taylor development is analytically continuated to the point p from one of the sheet along a curve on which no branch point lies, the analytic continuation from the other sheet reaches to p or the other point. The φ_t has a Taylor development at p:

$$\varphi^t = \left(\frac{\sqrt{c_1}}{2(w-1)^{\frac{3}{2}}} + ...\right)dw$$

and φ^t is holomorphic except for p over $w = 1$. These analytic continuations have different development at the reached points. This is a contradiction. Above all there is no moving branch point and hence the covering surface R_t is quite static.

Thus we have

Theorem 4. *Let a family R_t of finite compact bordered Riemann surfaces form a quasiconformal-holomorphic movement. If $\log \hat{K}_t(p)$ is harmonic with respect to t, every R_t is conformal to R_0.*

6 Robin constant

Let $G_t = G_t(,p)$ be a Green function on R_t with pole at $h_t(p)$ and set $\psi_p^t = dG_t + i * dG_t$. Robin constant at $h_t(p)$ is defined by

$$\gamma_t(p) = \frac{1}{2\pi i}\int_{|z|=\epsilon} G_t(z)\frac{dz}{z} + \log \epsilon.$$

Under a quasiconformal-holomorphic movement we have the following variational formulas[4].

Theorem 5.

$$\frac{\partial \gamma_t(p)}{\partial t} = \frac{1}{4\pi}(\frac{\partial \psi_p^t}{\partial t}, \overline{\psi_p^t}),$$
$$\frac{\partial^2 \gamma_t(p)}{\partial \bar{t}\partial t} = -\frac{1}{2\pi}(\frac{\partial \psi_p^t}{\partial \bar{t}}, \frac{\partial \psi_p^t}{\partial \bar{t}}).$$

Take a point q which is not zero point of ψ_p^t and assume that $\gamma_t(p)$ and $\gamma_t(q)$ are harmonic. Then

$$\frac{\partial \psi_p^t}{\partial \bar{t}} = 0, \quad \frac{\partial \psi_q^t}{\partial \bar{t}} = 0.$$

In the same way of Bergman kernel we have the following. The meromorphic function $w = \frac{\psi_p^t}{\psi_q^t}$ is pure imaginary along the boundary of R_t. Hence R_t is represented by w as a covering surface with slits over the imaginary axis. Since $w = w(z,t)$ is holomorphic with respect to t for a fixed z, these slits are static and only branch points may move as t varies. The point q lies on $w = 0$ and is not branch point. For $\psi_q^t = \Psi_q(w,t)dw$, $\Psi_q(w,t)$ is holomorphic with respect to t and w. Since ψ_q^t is pure imaginary along the boundary, $\Psi_q(w,t)$ is constant on the fixed boundary point on w with respect to t. Hence it is constant on every surface point on w with respect to t. If a branch point moves as t varies, then $\Psi_q(w,t)$ has the same Taylor development on the both sheets which contains the branch point. When the Taylor development is analytically continuated to the point q from one of the sheet along a curve on which no branch point lies, The analytic continuation from the other sheet reaches to a point except for q. The ψ_q^t is holomorphic except for q and the image q' of q by involution and has a singularity as $-\frac{1}{z}$ at q'. Their developments are different from the one at q. This contradicts to the result of analytic continuation. Above all there is no moving branch point and hence the covering surface R_t is quite static. Thus we have

Theorem 6. *Let R_t be a finite compact bordered Riemann surface with genus g which has m (> 0) boundary components. Let R_t forms a quasiconformal-holomorphic movement. For $2g + m + 1$ points $\{p_i\}$, if all $\gamma_t(p_i)$ are harmonic with respect to t, every R_t is conformal to R_0.*

$\psi_{p_1}^t$ is extended to its doubled Riemann surface and the total order of zero is $2(2g+m-1)$ and the total order of pole is 2. Since its total order of zero on R_t is at most $2g+m-1$, One of the points $\{p_i\}_{i=2,\ldots,2g+m+1}$ is not zero point of $\psi_{p_1}^t$. It follows the conclusion.

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