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THE ANALYTIC TORSIONS OF THE LINE BUNDLES OVER THE QUADRICS.

Chiaki Tsukamoto

Kyoto Institute of Technology
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Summary

The analytic torsions of the line bundles over the quadrics are computed directly from the defining spectral zeta functions associated with the Dolbeault complex. The spectral data needed are calculated using the branching rule for the symmetric pair $(SO(n+2), SO(2) \times SO(n))$ given by the author in [2]. The spectral zeta functions for the analytic torsions are shown to be of the form treated by the author in [3] and the derivatives at 0 are computed by the method developed there. The result is compared with the well-known Kai Köhler's paper [1]. The cancellation of the spectral zeta functions is observed, on the level of the spectral data.

Keywords: analytic torsion, branching rule, spectral zeta function

1. Introduction.

The analytic torsion of the Hermitian bundle over a complex Riemannian manifold is defined using the spectra of the Hodge Laplacians associated with the Dolbeault complex. The author gave the procedure to compute the spectra for Hermitian symmetric spaces in [3] and carried out the procedure for the projective spaces, which yields the direct computation of the analytic torsion of the line bundle. This procedure can be carried out for the quadrics, since the quadrics are Hermitian symmetric spaces $SO(n+2)/SO(2) \times SO(n)$.

We notice that $SO(n+2)/SO(2) \times SO(n)$ is of rank 2. This means that the spectral zeta functions are, in general, defined by a double sum. For example, the spectral zeta function for the space of smooth functions over the 3-dimensional quadric $SO(5)/SO(2) \times SO(3)$ is given by

$$\zeta_0(s) = -\frac{(12)^s}{3} \sum_{(k_1,k_2)\neq(0,0)} \frac{(4k_1+2k_2+3)(k_1+k_2+1)(2k_1+1)(2k_2+1)}{\{(2k_1+k_2)^2+3(2k_1+k_2)+k_2(k_2+1)\}^s}.$$

The analytic torsion is defined by the value of the differential at $s=0$ for the weighted sum of such spectral zeta functions. The value of the differential at $s=0$ for the above spectral zeta function seems hard to compute. But fortunately, in our case, there occurs a cancellation which yields a Dirichlet sum of usual type, and the analytic torsion is computable. Out main tool is the branching rule of representation for the pair $(SO(n+2), SO(2) \times SO(n))$ given in [2].

In fact, the cancellation procedure had been fully explained and carried out in the paper [1] by Kai Köhler, for any vector bundles over any Hermitian symmetric space.

In this paper, we shall give the very details that are specific for the case of quadrics, and obtain the concrete Dirichlet sums of the type treated in [3]. Finally, the analytic torsions are computed by the formula given in [3].

2. Hodge Laplacians on the quadrics.

The quadric Q^n is a submanifold of the complex projective space $P^{n+1}(\mathbf{C})$ given by the quadratic equation:

$$Q^n = \left\{ [z_1 : z_2 : \cdots : z_{n+2}] \in P^{n+2}(\mathbf{C}) \,\middle|\, \sum_{i=1}^{n+2} (z_i)^2 = 0 \right\}.$$

When we write the complex vector $\mathbf{z} = (z_1, z_2, \ldots, z_{n+2}) \in \mathbf{C}^{n+2}$ as a sum of two real vectors $\mathbf{x} = (x_1, x_2, \ldots, x_{n+2}) \in \mathbf{R}^{n+2}$ and $\mathbf{y} = (y_1, y_2, \ldots, y_{n+2}) \in \mathbf{R}^{n+2}$ like $\mathbf{z} = \mathbf{x} + \sqrt{-1}\mathbf{y}$, the condition $(\mathbf{z}, \mathbf{z}) = \sum_{i=1}^{n+2} (z_i)^2 = 0$ means that $(\mathbf{x}, \mathbf{x}) = (\mathbf{y}, \mathbf{y})$ and $(\mathbf{x}, \mathbf{y}) = 0$. Therefore $\mathbf{z}$ corresponds to an orthogonal basis $\mathbf{x}$, $\mathbf{y}$ of a 2-dimensional subspace of $\mathbf{R}^{n+2}$, and the class $[\mathbf{z}]$ corresponds to an oriented 2-dimensional subspace of $\mathbf{R}^{n+2}$. Thus we can indentify Q^n with the Grassmann manifold $SO(n+2)/SO(2) \times SO(n)$. We set $G = SO(n+2)$ and $K = SO(2) \times SO(n)$.

The Lie algebra $\mathfrak{g}$ of G is the space $so(n+2)$ of skew-symmetric matrices. It has an invariant inner product $B(X, Y) = -n \operatorname{trace} XY$ $(X,\, Y \in so(n+2))$, which is the Killing form sign-changed. With the Lie algebra $\mathfrak{k}$ of K, $\mathfrak{g}$ has an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. The restiction of B on $\mathfrak{m}$ gives the G-invariant Riemannian metric on Q^n.

We denote by $R(\theta)$ the 2-dimensional rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and by j the element $R(\pi/2) \times Id$ of K. The adjoint action $J = \operatorname{Ad}(j)$ on $\mathfrak{m}$ is the complex structure that corresponds to the complex structure of Q^n. We denote by $\mathfrak{m}_-$ [resp. $\mathfrak{m}_+$] the anti-holomorphic [resp. holomorphic] part of the complexification of $\mathfrak{m}$. We fix an orthonormal basis $\{E_a\}$ of $\mathfrak{m}_-$. The complex conjugate $\{\overline{E}_a\}$ forms an orthonormal basis of $\mathfrak{m}_+$, and satisfies $B(\overline{E}_a, E_{a'}) = \delta_{aa'}$. We also fix an orthonormal basis $\{F_b\}$ of $\mathfrak{k}$.

Let (χ, V_K) be an irreducible representation of K, where V_K is a complex vector space endowed with a K-invariant Hermitian inner product. We shall consider the associated vector bundle $E = G \times_K V_K$ on Q^n and the Dolbeault complex:

$$0 \to C^\infty(E) \xrightarrow{\overline{\partial}} C^\infty(E \otimes T^{0,1}Q^n) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C^\infty(E \otimes T^{0,n}Q^n).$$

We notice that $E \otimes T^{0,q}Q^n$ is also an associated vector bundle for the K-module $V_K \otimes \wedge^q(\mathfrak{m}_-)^*$, where we consider $\mathfrak{m}_-$ as a K-module by the adjoint action, and the dual space $(\mathfrak{m}_-)^*$ is isomorphic to $\mathfrak{m}_+$ as a K-module.

Since G/K is an Hermitian symmetric space, we can apply Theorem 1 of [2] to the Hodge Laplacian $\square$ on $C^\infty(E \otimes T^{0,q}Q^n)$.

Theorem 1. *We have*

$$\square = -\frac{1}{2}C_{\mathfrak{g}} + \frac{1}{2}(\chi(C_{\mathfrak{k}}) + \chi(R)),$$

where $C_{\mathfrak{g}}$ is the Casimir operator of $\mathfrak{g}$ given by

$$C_{\mathfrak{g}} = \sum_a (\overline{E}_a E_a + E_a \overline{E}_a) + \sum_b Y_b Y_b,$$

and $\chi(C_{\mathfrak{k}})$ and $\chi(R)$ are defined using the Lie algebra action χ of $\mathfrak{k}$:

$$\chi(C_{\mathfrak{k}}) = \sum_b \chi(F_b)\chi(F_b), \qquad \chi(R) = \sum_a \chi([\overline{E}_a, E_a]).$$

We notice that $\chi(C_{\mathfrak{k}})$ and $\chi(R)$ are constants depending only on (χ, V_K).

The space of smooth sections $C^\infty(E \otimes T^{0,q}Q^n)$ decomposes into the sum of irreducible G-submodules. We denote by $V_G(\Lambda_G)$ the irreducible G-module with the highest weight Λ_G. The action of $C_{\mathfrak{g}}$ on the submodule isomorphic to $V_G(\Lambda_G)$ is given by the Freudenthal's formula $-B(\Lambda_G + 2\delta_G, \Lambda_G)$, where δ_G is the half of the sum of the positive roots of $\mathfrak{g}$. The action of $C_{\mathfrak{k}}$ on (χ, V_K) is computed by the same formula. When we denote by Λ_K the highest weight of (χ, V_K), the action of $\chi(C_{\mathfrak{k}})$ is given by $-B(\Lambda_K + 2\delta_K, \Lambda_K)$, where δ_K is the half of the sum of the positive roots of $\mathfrak{k}$.

For an Hermitian symmetric space G/K, the weights of the K-module $\mathfrak{m}_+$ coincides with the collection of positive roots of $\mathfrak{g}$ other than positive roots of $\mathfrak{k}$. We denote by Δ_+^G [resp. Δ_+^K] the positive roots of $\mathfrak{g}$ [resp. $\mathfrak{k}$]. We may assume that $\overline{E}_a$ are the eigenvectors associated with the positive roots $\alpha \in \Delta_+^G \setminus \Delta_+^K$. Then E_a are the eigenvectors associated with the negative roots $-\alpha$. We can show that

$$R = \sum_a [\overline{E}_a, E_a] = -\sum_{\alpha \in \Delta_+^G \setminus \Delta_+^K} H_\alpha,$$

where H_α is the dual vector of the positive root α, from which we can deduce that $\chi(R) = -B(2\delta_G - 2\delta_K, \Lambda_K)$. Theorem 1 is rewritten as follows:

Corollary 2. *The action of $\square$ on the irreducible G-submodule of $C^\infty(E \otimes T^{0,q}Q^n)$ isomorphic to $V_G(\Lambda_G)$ is the multiplication of the constant $\mu(\Lambda_G)$ given by*

$$\mu(\Lambda_G) = \frac{1}{2}\left(B(\Lambda_G + 2\delta_G, \Lambda_G) - B(\Lambda_K + 2\delta_G, \Lambda_K)\right).$$

Thus we get an eigenvalue $\mu(\Lambda_G)$ of $\square$, which coincides with the one given in Lemma 17 of [1]. The multiplicity of the eigenvalue is the product of the multiplicity $m(q, \Lambda_G)$ of the G-module $V_G(\Lambda_G)$ in the decomposition $C^\infty(E \otimes T^{0,q}Q^n) \cong \sum_{\Lambda_G} m(q, \Lambda_G) V_G(\Lambda_G)$ and its dimension $\dim V_G(\Lambda_G)$.

To compute the analytic torsion A-Tor(E), we make the spectral zeta functions $\zeta_{E,q}(s)$ for $C^\infty(E \otimes T^{0,q}Q^n)$ and $\zeta_E(s)$:

$$\zeta_{E,q}(s) = \sum_{\Lambda_G} m(q, \Lambda_G) \dim V_G(\Lambda_G) \left(\mu(\Lambda_G)\right)^{-s},$$

$$\zeta_E(s) = \sum_{q=0}^{n} (-1)^q \, q \, \zeta_{E,q}(s).$$

The analytic continuation of these functions are holomorphic in the neighborhood of $s = 0$, and the analytic torsion is defined by A-Tor$(E) = \zeta_E'(0)$.

Notice that $\mu(\Lambda_G)$ does not depend on q. We set

$$D(s, \Lambda_G) = \dim V_G(\Lambda_G)\, (\mu(\Lambda_G))^{-s} ,$$

and then we have

$$\begin{aligned}\zeta_E(s) &= \sum_{q=0}^{n}(-1)^q\, q \sum_{\Lambda_G} m(q, \Lambda_G) D(s, \Lambda_G) \\ &= \sum_{\Lambda_G}\left(\sum_{q=0}^{n}(-1)^q\, q \cdot m(q, \Lambda_G)\right) D(s, \Lambda_G).\end{aligned}$$

We fix a maximal torus $T \subset K \subset G$, and denote by W_G [resp. W_K] the Weyl group of G [resp. K], which acts on the Lie algebra of T, and on the space of weights of G-modules [resp. K-modules]. We denote by $\chi_G(\Lambda_G)$ the character of the irreducible G-module with the highest weight Λ_G.

The main observation of Kai Köhler [1] is that, by the orthogonality of characters of K-modules, $m(q, \Lambda_G)$ is computed by the integral over T of the product of $\chi_G(\Lambda_G)$ and the conjugate of the character of $V_K \otimes \wedge^q \mathfrak{m}_+$. Therefore, $\sum_{q=0}^{n}(-1)^q\, q \cdot m(q, \Lambda_G)$ can be computed by the integral of the product of $\chi_G(\Lambda_G)$ and the conjugate of the virtual character of the virtual K-module $V_K \otimes \sum_{q=0}^{n}(-1)^q\, q \cdot \wedge^q \mathfrak{m}_+$. Moreover, since the set of the weights of $\mathfrak{m}_+$ is just the set of the positive roots in $\Delta_+^G \setminus \Delta_+^K$, the virtual character has an explicit representation. Using the Weyl character formula, Kai Köhler [1] proved the following theorem:

Theorem 3. *The spectral zeta function $\zeta_E(s)$ is given by*

$$\begin{aligned}\zeta_E(s) = -2^s \sum_{\alpha \in \Delta_+^G \setminus \Delta_+^K} \sum_{k=1}^{\infty} &\operatorname{sgn}(w) \dim V_G(w(\delta_G + \Lambda_K + k\alpha) - \delta_G) \\ &\times B(k\alpha, k\alpha + 2\delta_G + 2\Lambda_K)^{-s},\end{aligned}$$

where the sum is in fact taken over the pair (α, k) such that, for some $w \in W_G$, $w(\delta_G + \Lambda_K + k\alpha) - \delta_G$ is the highest weight of an irreducible G-module.

See Lemma 4 and the formula (115) in the section 11 of [1].

This theorem gives an account for the cancellation and the general resemblance of the spectral zeta function, but, to compute the concrete value of the analytic torsion, we need some more effort.

We shall give the spectral zeta function directly for the 1-dimensional representation $(\chi_p, \mathbf{C})$ of K that is given by $\chi_p(R(\theta) \times S) = \exp(\sqrt{-1}\, p\, \theta)$ for an integer p. We denote the corresponding line bundle by L^p. For $p = -1$, L^p is the restriction of the tautological line bundle over $P^{n+1}(\mathbf{C})$ to Q^n. We assume $p \geq 0$ and devide the cases for $n = 2m$ $(m \geq 2)$ and $n = 2m + 1$ $(m \geq 1)$.

3. Computation for $SO(2m+2)/SO(2) \times SO(2m)$.

We take the maximal torus $T = \{ R(\theta_0, \theta_1, \dots, \theta_m) = R(\theta_0) \times R(\theta_1) \times \cdots \times R(\theta_m) \}$ for both G and K. We take a basis $\lambda_0, \lambda_1, \dots, \lambda_m$ for the characters, where λ_k $(0 \le k \le m)$ satisfies $\exp(\lambda_k(R(\theta_0, \theta_1, \dots, \theta_m))) = \exp(\sqrt{-1}\theta_k)$.

The K-module $(\chi_p, \mathbf{C})$ has the highest weight $p\lambda_0$. The K-module $V_K^q = (\chi_p, \mathbf{C}) \otimes \wedge^q(\mathfrak{m}_-)^*$ is the irreducible K-module with the highest weight $(p+q)\lambda_0 + \lambda_1 + \cdots + \lambda_q$ for $0 < q < m$, is the sum of the irreducible K-modules with the highest weights $(p+m)\lambda_0 + \lambda_1 + \cdots + \lambda_{m-1} \pm \lambda_m$ for $q = m$, and is the irreducible K-module with the highest weight $(p+q)\lambda_0 + \lambda_1 + \cdots + \lambda_{2m-q}$ for $m < q < 2m$, and with the highest weight $(p+2m)\lambda_0$ for $q = 2m$.

An irreducible G-module has the highest weight of the form $\Lambda_G = h_0\lambda_0 + h_1\lambda_1 + \cdots + h_{m-1}\lambda_{m-1} \pm h_m\lambda_m$, where $h_0, h_1, \dots, h_{m-1}$, and h_m are integers satisfying $h_0 \ge h_1 \ge \cdots \ge h_{m-1} \ge h_m \ge 0$. We set $\Lambda_q = \sum_{i=0}^{q} \lambda_i$ for $1 \le q < m$ and $\Lambda_m^{\pm} = \Lambda_{m-1} \pm \lambda_m$.

The decomposion of $C^\infty(L^p \otimes T^{0,q}Q^n)$ into the irreducible G-modules can be determined by examining in which decomposition of irreducible G-module $V_G(\Lambda_G)$ the K-module V_K^q appears, which can be carried out by the branching rule given in [2]. We give Λ_G for which $V_G(\Lambda_G)$ appears in the decomposition of $C^\infty(L^p \otimes T^{0,q}Q^n)$, and the multiplicity $m(q, \Lambda_G)$ in the following Table 1:

q	Λ_G	$m(q,\Lambda_G)$
0	$(2k_1+p)\lambda_0 + k_2\Lambda_1$	1
1	$(2k_1+p)\lambda_0 + k_2\Lambda_1$	1 for $k_1 = 0$ and $k_2 \ge 1$ 1 for $k_1 \ge 1$ and $k_2 = 0$ 2 for $k_1 \ge 1$ and $k_2 \ge 1$
	$(2k_1+p+1)\lambda_0 + k_2\Lambda_1 + \Lambda_2$	1
2	$(2k_1+p)\lambda_0 + k_2\Lambda_1$	1 for $k_1 \ge 1$ and $k_2 \ge 1$
	$(2k_1+p+1)\lambda_0 + k_2\Lambda_1 + \Lambda_2$	1 for $k_1 = 0$ and $k_2 \ge 0$ 2 for $k_1 \ge 1$ and $k_2 \ge 0$
	$(2k_1+p+2)\lambda_0 + k_2\Lambda_1 + \Lambda_3$	1
$(*)$	$(2k_1+p+q-2)\lambda_0 + k_2\Lambda_1 + \Lambda_{q-1}$	1 for $k_1 \ge 1$ and $k_2 \ge 0$
	$(2k_1+p+q-1)\lambda_0 + k_2\Lambda_1 + \Lambda_q$	1 for $k_1 = 0$ and $k_2 \ge 0$ 2 for $k_1 \ge 1$ and $k_2 \ge 0$
	$(2k_1+p+q)\lambda_0 + k_2\Lambda_1 + \Lambda_{q+1}$	1
$m-1$	$(2k_1+p+m-3)\lambda_0 + k_2\Lambda_1 + \Lambda_{m-2}$	1 for $k_1 \ge 1$ and $k_2 \ge 0$
	$(2k_1+p+m-2)\lambda_0 + k_2\Lambda_1 + \Lambda_{m-1}$	1 for $k_1 = 0$ and $k_2 \ge 0$ 2 for $k_1 \ge 1$ and $k_2 \ge 0$
	$(2k_1+p+m-1)\lambda_0 + k_2\Lambda_1 + \Lambda_m^+$	1
	$(2k_1+p+m-1)\lambda_0 + k_2\Lambda_1 + \Lambda_m^-$	1

Table 1 (Part 1): Highest weight Λ_G and its multiplicity $m(q, \Lambda_G)$ for $n = 2m$. $((*)\colon 2 < q < m-1.)$

q	Λ_G	$m(q,\Lambda_G)$
m	$(2k_1+p+m-2)\lambda_0+k_2\Lambda_1+\Lambda_{m-1}$	2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+m-1)\lambda_0+k_2\Lambda_1+\Lambda_m^+$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+m-1)\lambda_0+k_2\Lambda_1+\Lambda_m^-$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
$m+1$	$(2k_1+p+m-1)\lambda_0+k_2\Lambda_1+\Lambda_m^+$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+m-1)\lambda_0+k_2\Lambda_1+\Lambda_m^-$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+m)\lambda_0+k_2\Lambda_1+\Lambda_{m-1}$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 0$ and $k_2 \geq 0$
	$(2k_1+p+m+1)\lambda_0+k_2\Lambda_1+\Lambda_{m-2}$	1
$(**)$	$(2k_1+p+q-2)\lambda_0+k_2\Lambda_1+\Lambda_{2m-q+1}$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+q-1)\lambda_0+k_2\Lambda_1+\Lambda_{2m-q}$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+q)\lambda_0+k_2\Lambda_1+\Lambda_{2m-q-1}$	1
$2m-2$	$(2k_1+p+2m-4)\lambda_0+k_2\Lambda_1+\Lambda_3$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+2m-3)\lambda_0+k_2\Lambda_1+\Lambda_2$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+2m-2)\lambda_0+k_2\Lambda_1$	1 for $k_1 \geq 0$ and $k_2 \geq 1$
$2m-1$	$(2k_1+p+2m-3)\lambda_0+k_2\Lambda_1+\Lambda_2$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1+p+2m-2)\lambda_0+k_2\Lambda_1$	1 for $k_1 = 0$ and $k_2 \geq 1$ 1 for $k_1 \geq 1$ and $k_2 = 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 1$
$2m$	$(2k_1+p+2m-2)\lambda_0+k_2\Lambda_1$	1 for $k_1 \geq 1$ and $k_2 \geq 0$

Table 1 (Part 2): Highest weight Λ_G and its multiplicity $m(q,\Lambda_G)$ for $n=2m$. ($(**)$: $m+1<q<2m-2$.)

To see what happens for each $V_G(\Lambda_G)$, we rearrange Table 1 to the next Table 2.

Λ_G	$(-1)^q\, q$	$m(q,\Lambda_G)$
$(2k_1+p)\lambda_0+k_2\Lambda_1$	0	1
	-1	1 for $k_1 = 0$ and $k_2 \geq 1$ 1 for $k_1 \geq 1$ and $k_2 = 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 1$
	2	1 for $k_1 \geq 1$ and $k_2 \geq 1$

Table 2 (Part 1): The multiplicity $m(q,\Lambda_G)$ for $n=2m$.

Λ_G	$(-1)^q\, q$	$m(q, \Lambda_G)$
	$(-1)^{q-1}\,(q-1)$	1
$(2k_1 + p + q - 1)\lambda_0 + k_2\Lambda_1 + \Lambda_q$	$(-1)^q\, q$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
$(2 \leq q \leq m-1)$	$(-1)^{q+1}\,(q+1)$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(-1)^{m-1}\,(m-1)$	1
$(2k_1 + p + m - 1)\lambda_0 + k_2\Lambda_1 + \Lambda_m^{\pm}$	$(-1)^m\, m$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(-1)^{m+1}\,(m+1)$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(-1)^{q-1}\,(q-1)$	1
$(2k_1 + p + q - 1)\lambda_0 + k_2\Lambda_1$ $+\Lambda_{2m-q}$	$(-1)^q\, q$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
$(m+1 \leq q \leq 2m-2)$	$(-1)^{q+1}\,(q+1)$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$2m-2$	1 for $k_1 \geq 0$ and $k_2 \geq 1$
$(2k_1 + p + 2m - 2)\lambda_0 + k_2\Lambda_1$	$-(2m-1)$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$2m$	1 for $k_1 \geq 1$ and $k_2 \geq 0$

Table 2 (Part 2): The multiplicity $m(q, \Lambda_G)$ for $n = 2m$.

Taking account of the cancellation of terms, we get the following:

Proposition 4. *The spectral zeta function $\zeta_{L^p}(s)$ is given by*

$$\begin{aligned}
\zeta_{L^p}(s) = &-\sum_{k=1}^{\infty} D(s, (2k+p)\lambda_0) \\
&+ \sum_{q=1}^{m-1} (-1)^q \sum_{k=0}^{\infty} D(s, (p+q-1)\lambda_0 + k\Lambda_1 + \Lambda_q) \\
&+ (-1)^m \sum_{k=0}^{\infty} D(s, (p+m-1)\lambda_0 + k\Lambda_1 + \Lambda_m^+) \\
&+ (-1)^m \sum_{k=0}^{\infty} D(s, (p+m-1)\lambda_0 + k\Lambda_1 + \Lambda_m^-) \\
&+ \sum_{q=m+1}^{2m-1} (-1)^q \sum_{k=0}^{\infty} D(s, (p+q-1)\lambda_0 + k\Lambda_1 + \Lambda_{2m-q}) \\
&+ \sum_{k=0}^{\infty} D(s, (2k+p+2m)\lambda_0).
\end{aligned}$$

We notice that the sum of the first term and the last term reduce to the finite sum.

$$
\begin{aligned}
-\sum_{k=1}^{\infty} D(s,(2k+p)\lambda_0) &+ \sum_{k=0}^{\infty} D(s,(2k+p+2m)\lambda_0) \\
&= -\sum_{k=1}^{m-1} D(s,(2k+p)\lambda_0).
\end{aligned}
$$

Proposition 4 agrees with the result given by Theorem 3.

The value of $\mu(\Lambda_G)$ is given by

$$
\begin{aligned}
\mu(\Lambda_G) &= \frac{1}{2}\left(B(\Lambda_G+2\delta_G,\Lambda_G)+\chi_p(C_{\mathfrak{k}})+\chi_p(R)\right) \\
&= \frac{1}{2}\left(B(\Lambda_G+2\delta_G,\Lambda_G)-B(p\lambda_0+2\delta_G,p\lambda_0)\right),
\end{aligned}
$$

which can be computed using

$$
\delta_G = \sum_{i=0}^{m-1}(m-i)\lambda_i, \qquad B(\lambda_i,\lambda_j) = \frac{1}{4m}\delta_{ij}.
$$

Proposition 5. *$\mu(\Lambda_G)$ are given as follows:*

$$
\begin{aligned}
\mu((2k+p)\lambda_0) &= \frac{1}{2m}k(k+p+m), \\
\mu((p+q-1)\lambda_0+k\Lambda_1+\Lambda_q) &= \frac{1}{4m}(k+q)(k+p+2m) \\
&\qquad (1 \le q \le m-1), \\
\mu((p+m-1)\lambda_0+k\Lambda_1+\Lambda_m^{\pm}) &= \frac{1}{4m}(k+m)(k+p+2m), \\
\mu((p+q-1)\lambda_0+k\Lambda_1+\Lambda_{2m-q}) &= \frac{1}{4m}(k+q)(k+p+2m) \\
&\qquad (m+1 \le q \le 2m-1).
\end{aligned}
$$

The value of $\dim V_G(\Lambda_G)$ is given by the Weyl dimension formula:

$$
\dim V_G(\Lambda_G) = \prod_{\alpha\in\Delta_+^G} \frac{B(\Lambda_G+\delta_G,\alpha)}{B(\delta_G,\alpha)},
$$

where the product is taken over all the positive roots $\alpha = \lambda_i+\lambda_j$ $(0 \le i < j \le m)$, $\lambda_i-\lambda_j$ $(0 \le i < j \le m)$.

Proposition 6. $\dim V_G(\Lambda_G)$ *are given as follows:*

$$
\begin{aligned}
&\dim V_G((2k+p)\lambda_0) \\
&\quad = \frac{4k+2p+2m}{2m}\binom{2k+p+2m-1}{2m-1}, \\
&\dim V_G(p+q-1)\lambda_0 + k\Lambda_1 + \Lambda_q) \\
&\quad = \frac{2(p+q)q}{m}\binom{2m-1}{q} \times \frac{(2k+p+q+2m)\cdot(k+p+q+m)(k+m)}{(k+p+2m)(k+q)\cdot(k+p+2q)(k+2m-q)} \\
&\quad\quad \times \binom{k+p+q+2m-1}{2m-1}\binom{k+2m-1}{2m-1} \qquad (1 \le q \le m-1), \\
&\dim V_G((p+m-1)\lambda_0 + k\Lambda_1 + \Lambda_m^{\pm}) \\
&\quad = (p+m)\binom{2m-1}{m} \times \frac{(2k+p+3m)}{(k+p+2m)(k+m)} \\
&\quad\quad \times \binom{k+p+3m-1}{2m-1}\binom{k+2m-1}{2m-1}, \\
&\dim V_G((p+q-1)\lambda_0 + k\Lambda_1 + \Lambda_{2m-q}) \\
&\quad = \frac{2(p+q)q}{m}\binom{2m-1}{q} \times \frac{(2k+p+q+2m)\cdot(k+p+q+m)(k+m)}{(k+p+2m)(k+q)\cdot(k+p+2q)(k+2m-q)} \\
&\quad\quad \times \binom{k+p+q+2m-1}{2m-1}\binom{k+2m-1}{2m-1} \qquad (m+1 \le q \le 2m-1).
\end{aligned}
$$

We notice that all the $\dim V_G(\Lambda_G)$ are polynomials in k.
Thus we get the following explicit formula:

Theorem 7. *The spectral zeta function $\zeta_{L^p}(s)$ is given by*

$$
\begin{aligned}
\zeta_{L^p}(s) = &-\sum_{k=1}^{m-1} \frac{4k+2p+2m}{2m}\binom{2k+p+2m-1}{2m-1}\left\{\frac{1}{2m}k(k+p+m)\right\}^{-s} \\
&+ \sum_{q=1}^{2m-1} (-1)^q \frac{2(p+q)q}{m}\binom{2m-1}{q} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(2k+p+q+2m)\cdot(k+p+q+m)(k+m)}{(k+p+2m)(k+q)\cdot(k+p+2q)(k+2m-q)} \\
&\quad\quad \times \binom{k+p+q+2m-1}{2m-1}\binom{k+2m-1}{2m-1} \\
&\quad\quad \times \left\{\frac{1}{4m}(k+q)(k+p+2m)\right\}^{-s}.
\end{aligned}
$$

Since the Dirichlet sums appearing in the above formula are of the type treated in [3], the analytic torsion A-Tor(L^p) can be computed by Theorems 6 and 8 of [3].

4. Computation for $SO(2m+3)/SO(2)\times SO(2m+1)$.

We take the maximal torus $T=\{\,R(\theta_0,\theta_1,\dots,\theta_m)=R(\theta_0)\times R(\theta_1)\times\cdots\times R(\theta_m)\times 1\,\}$ for both G and K. We take a basis $\lambda_0,\ \lambda_1,\ \dots,\ \lambda_m$ for the characters, where λ_k $(0\le k\le m)$ satisfies $\exp(\lambda_k(R(\theta_0,\theta_1,\dots,\theta_m)))=\exp(\sqrt{-1}\theta_k)$.

The K-module $(\chi_p,\mathbf{C})$ has the highest weight $p\lambda_0$. The K-module $V_K^q=(\chi_p,\mathbf{C})\otimes\wedge^q(\mathfrak{m}_-)^*$ is the irreducible K-module with the highest weight $(p+q)\lambda_0+\lambda_1+\cdots+\lambda_q$ for $0<q\le m$, with the highest weight $(p+q)\lambda_0+\lambda_1+\cdots+\lambda_{2m-q}$ for $m+1\le q<2m$, and with the highest weight $(p+2m)\lambda_0$ for $q=2m$.

An irreducible G-module has the highest weight of the form $\Lambda_G=h_0\lambda_0+h_1\lambda_1+\cdots+h_m\lambda_m$, where $h_0,\ h_1,\ \dots$, and h_m are integers satisfying $h_0\ge h_1\ge\cdots\ge h_m\ge 0$. We set $\Lambda_q=\sum_{i=0}^{q}\lambda_i$ for $1\le q\le m$.

We give Λ_G for which $V_G(\Lambda_G)$ appears in the decomposition of $C^\infty(L^p\otimes T^{0,q}Q^n)$, and the multiplicity $m(q,\Lambda_G)$ in the following Table 3 and Table 4:

q	Λ_G	$m(q,\Lambda_G)$
0	$(2k_1+p)\lambda_0+k_2\Lambda_1$	1
1	$(2k_1+p)\lambda_0+k_2\Lambda_1$	1 for $k_1=0$ and $k_2\ge 1$ 1 for $k_1\ge 1$ and $k_2=0$ 2 for $k_1\ge 1$ and $k_2\ge 1$
	$(2k_1+p+1)\lambda_0+k_2\Lambda_1+\Lambda_2$	1
2	$(2k_1+p)\lambda_0+k_2\Lambda_1$	1 for $k_1\ge 1$ and $k_2\ge 1$
	$(2k_1+p+1)\lambda_0+k_2\Lambda_1+\Lambda_2$	1 for $k_1=0$ and $k_2\ge 0$ 2 for $k_1\ge 1$ and $k_2\ge 0$
	$(2k_1+p+2)\lambda_0+k_2\Lambda_1+\Lambda_3$	1
$(*)$	$(2k_1+p+q-2)\lambda_0+k_2\Lambda_1+\Lambda_{q-1}$	1 for $k_1\ge 1$ and $k_2\ge 0$
	$(2k_1+p+q-1)\lambda_0+k_2\Lambda_1+\Lambda_q$	1 for $k_1=0$ and $k_2\ge 0$ 2 for $k_1\ge 1$ and $k_2\ge 0$
	$(2k_1+p+q)\lambda_0+k_2\Lambda_1+\Lambda_{q+1}$	1
m	$(2k_1+p+m-2)\lambda_0+k_2\Lambda_1+\Lambda_{m-1}$	1 for $k_1\ge 1$ and $k_2\ge 0$
	$(2k_1+p+m-1)\lambda_0+k_2\Lambda_1+\Lambda_m$	1 for $k_1=0$ and $k_2\ge 0$ 2 for $k_1\ge 1$ and $k_2\ge 0$
	$(2k_1+p+m)\lambda_0+k_2\Lambda_1+\Lambda_m$	1
$m+1$	$(2k_1+p+m-1)\lambda_0+k_2\Lambda_1+\Lambda_m$	1 for $k_1\ge 1$ and $k_2\ge 0$
	$(2k_1+p+m)\lambda_0+k_2\Lambda_1+\Lambda_m$	1 for $k_1=0$ and $k_2\ge 0$ 2 for $k_1\ge 1$ and $k_2\ge 0$
	$(2k_1+p+m+1)\lambda_0+k_2\Lambda_1+\Lambda_{m-1}$	1

Table 3 (Part 1): Highest weight Λ_G and its multiplicity $m(q,\Lambda_G)$ for $n=2m+1$. $((*)\colon 2<q<m.)$

q	Λ_G	$m(q, \Lambda_G)$
$(**)$	$(2k_1 + p + q - 2)\lambda_0 + k_2\Lambda_1 + \Lambda_{2m-q+2}$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1 + p + q - 1)\lambda_0 + k_2\Lambda_1 + \Lambda_{2m-q+1}$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1 + p + q)\lambda_0 + k_2\Lambda_1 + \Lambda_{2m-q}$	1
$2m - 1$	$(2k_1 + p + 2m - 3)\lambda_0 + k_2\Lambda_1$	1 for $k_1 \geq 1$ and $k_2 \geq 1$
	$(2k_1 + p + 2m - 2)\lambda_0 + k_2\Lambda_1 + \Lambda_2$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1 + p + 2m - 1)\lambda_0 + k_2\Lambda_1 + \Lambda_3$	1
$2m$	$(2k_1 + p + 2m - 2)\lambda_0 + k_2\Lambda_1 + \Lambda_2$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(2k_1 + p + 2m - 1)\lambda_0 + k_2\Lambda_1$	1 for $k_1 = 0$ and $k_2 \geq 1$ 1 for $k_1 \geq 1$ and $k_2 = 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 1$
$2m + 1$	$(2k_1 + p + 2m - 1)\lambda_0 + k_2\Lambda_1$	1 for $k_1 \geq 1$ and $k_2 \geq 0$

Table 3 (Part 2): Highest weight Λ_G and its multiplicity $m(q, \Lambda_G)$ for $n = 2m + 1$. ($(**)$: $m + 1 < q < 2m - 1$.)

Λ_G	$(-1)^q\, q$	$m(q, \Lambda_G)$
$(2k_1 + p)\lambda_0 + k_2\Lambda_1$	0	1
	-1	1 for $k_1 = 0$ and $k_2 \geq 1$ 1 for $k_1 \geq 1$ and $k_2 = 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 1$
	2	1 for $k_1 \geq 1$ and $k_2 \geq 1$
$(2k_1 + p + q - 1)\lambda_0 + k_2\Lambda_1 + \Lambda_q$ $(2 \leq q \leq m)$	$(-1)^{q-1}\,(q-1)$	1
	$(-1)^q\, q$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(-1)^{q+1}\,(q+1)$	1 for $k_1 \geq 1$ and $k_2 \geq 0$
$(2k_1 + p + q - 1)\lambda_0 + k_2\Lambda_1 + \Lambda_{2m-q+1}$ $(m + 1 \leq q \leq 2m - 1)$	$(-1)^{q-1}\,(q-1)$	1
	$(-1)^q\, q$	1 for $k_1 = 0$ and $k_2 \geq 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 0$
	$(-1)^{q+1}\,(q+1)$	1 for $k_1 \geq 1$ and $k_2 \geq 0$

Table 4 (Part 1): The multiplicity $m(q, \Lambda_G)$ for $n = 2m + 1$.

Λ_G	$(-1)^q\, q$	$m(q, \Lambda_G)$
$(2k_1 + p + 2m - 1)\lambda_0 + k_2\Lambda_1$	$-(2m-1)$	1 for $k_1 \geq 0$ and $k_2 \geq 1$
	$2m$	1 for $k_1 = 0$ and $k_2 \geq 1$ 1 for $k_1 \geq 1$ and $k_2 = 0$ 2 for $k_1 \geq 1$ and $k_2 \geq 1$
	$-(2m+1)$	1 for $k_1 \geq 1$ and $k_2 \geq 0$

Table 4 (Part 2): The multiplicity $m(q, \Lambda_G)$ for $n = 2m + 1$.

Taking account of the cancellation of terms, we get the following, which agrees with Theorem 3.

Proposition 8. *The spectral zeta function $\zeta_{L^p}(s)$ is given by*

$$\begin{aligned}\zeta_{L^p}(s) = &-\sum_{k=1}^{\infty} D(s, (2k+p)\lambda_0) - \sum_{k=0}^{\infty} D(s, (2k+p+2m+1)\lambda_0) \\ &+ \sum_{q=1}^{m} (-1)^q \sum_{k=0}^{\infty} D(s, (p+q-1)\lambda_0 + k\Lambda_1 + \Lambda_q) \\ &+ \sum_{q=m+1}^{2m} (-1)^q \sum_{k=0}^{\infty} D(s, (p+q-1)\lambda_0 + k\Lambda_1 + \Lambda_{2m-q+1}).\end{aligned}$$

We compute the value of $\mu(\Lambda_G)$ using

$$\delta_G = \sum_{i=0}^{m} \left(m - i + \frac{1}{2}\right)\lambda_i, \qquad B(\lambda_i, \lambda_j) = \frac{1}{2(2m+1)}\delta_{ij}.$$

Proposition 9. *$\mu(\Lambda_G)$ are given as follows:*

$$\begin{aligned}\mu((p+2k)\lambda_0) &= \frac{1}{4(2m+1)} 2k(2k+2p+2m+1), \\ \mu((p+q-1)\lambda_0 + k\Lambda_1 + \Lambda_q) &= \frac{1}{2(2m+1)}(k+q)(k+p+2m+1) \\ &\qquad\qquad (1 \leq q \leq m), \\ \mu((p+q-1)\lambda_0 + k\Lambda_1 + \Lambda_{2m+1-q}) &= \frac{1}{2(2m+1)}(k+q)(k+p+2m+1) \\ &\qquad\qquad (m+1 \leq q \leq 2m).\end{aligned}$$

In this case, the positive roots are $\lambda_i + \lambda_j$ $(0 \leq i < j \leq m)$, $\lambda_i - \lambda_j$ $(0 \leq i < j \leq m)$, and λ_i $(0 \leq i \leq m)$.

Proposition 10. $\dim V_G(\Lambda_G)$ *are given as follows:*

$$\begin{aligned}
&\dim V_G((2k+p)\lambda_0)\\
&\quad= \frac{4k+2p+2m+1}{2m+1}\binom{2k+p+2m}{2m},\\
&\dim V_G(p+q-1)\lambda_0+k\Lambda_1+\Lambda_q)\\
&\quad= \frac{4(p+q)q}{2m+1}\binom{2m}{q}\times\frac{(2k+p+q+2m+1)\cdot\left(k+p+q+m+\frac{1}{2}\right)\left(k+m+\frac{1}{2}\right)}{(k+p+2m+1)(k+q)\cdot(k+p+2q)(k+2m+1-q)}\\
&\quad\times\binom{k+p+q+2m}{2m}\binom{k+2m}{2m}\qquad(1\le q\le m),\\
&\dim V_G((p+q-1)\lambda_0+k\Lambda_1+\Lambda_{2m+1-q})\\
&\quad= \frac{4(p+q)q}{2m+1}\binom{2m}{q}\times\frac{(2k+p+q+2m+1)\cdot\left(k+p+q+m+\frac{1}{2}\right)\left(k+m+\frac{1}{2}\right)}{(k+p+2m+1)(k+q)\cdot(k+p+2q)(k+2m+1-q)}\\
&\quad\times\binom{k+p+q+2m}{2m}\binom{k+2m}{2m}\qquad(m+1\le q\le 2m).
\end{aligned}$$

We again notice that all the $\dim V_G(\Lambda_G)$ are polynomials in k.
Thus we get the following:

Theorem 11. *The spectral zeta function* $\zeta_{L^p}(s)$ *is given by*

$$\begin{aligned}
\zeta_{L^p}(s) = &-\sum_{k=1}^{m-1}\frac{4k+2p+2m+1}{2m+1}\binom{2k+p+2m}{2m}\\
&\qquad\times\left\{\frac{1}{4(2m+1)}2k(2k+2p+2m+1)\right\}^{-s}\\
&-\sum_{k=0}^{\infty}\frac{2k+2p+6m+1}{2m+1}\binom{k+p+4m}{2m}\\
&\qquad\times\left\{\frac{1}{4(2m+1)}(k+2m)(k+2p+4m+1)\right\}^{-s}\\
&+\sum_{q=1}^{2m}(-1)^q\frac{4(p+q)q}{2m+1}\binom{2m}{q}\\
&\qquad\times\sum_{k=0}^{\infty}\frac{(2k+p+q+2m+1)\cdot\left(k+p+q+m+\frac{1}{2}\right)\left(k+m+\frac{1}{2}\right)}{(k+p+2m+1)(k+q)\cdot(k+p+2q)(k+2m+1-q)}\\
&\qquad\quad\times\binom{k+p+q+2m}{2m}\binom{k+2m}{2m}\\
&\qquad\quad\times\left\{\frac{1}{2(2m+1)}(k+q)(k+p+2m+1)\right\}^{-s}.
\end{aligned}$$

Since the Dirichlet sums appearing in the above formula are of the type treated in [3] too, the analytic torsion A-Tor(L^p) can again be computed by Theorems 6 and 8 of [3].

5. Examples.

In §3, we omitted the case $n = 2$. The quadric $Q^2 = SO(4)/SO(2)\times SO(2)$ is isomorphic to $P^1(\mathbf{C}) \times P^1(\mathbf{C})$. The computation is mostly the same, and we get

$$\begin{aligned}\zeta_{L^p}(s) &= -2(p+1)\sum_{k=0}^{\infty}(2k+p+3)\left\{\frac{1}{4}(k+1)(k+p+2)\right\}^{-s} \\ &= -4\cdot 4^s \cdot (p+1)\sum_{k=0}^{\infty}\left(k+\frac{p+3}{2}\right)\{(k+1)(k+p+2)\}^{-s}.\end{aligned}$$

In general, for $0 < \delta_1 < \delta_2$, we set

$$\delta_+ = \frac{\delta_1+\delta_2}{2}, \qquad \delta_- = \frac{\delta_2-\delta_1}{2},$$

and, for a polynomial $P(k)$ which is odd with respect to $k = -\delta_+$, we define the Dirichlet sum $\zeta(P(k), \delta_1, \delta_2; s)$ by

$$\zeta(P(k),\delta_1,\delta_2;s) = \sum_{k=0}^{\infty} P(k)\{(k+\delta_1)(k+\delta_2)\}^{-s}.$$

For $\delta_1 = 1$, $\delta_2 = p+2$, and $P(k) = k + \delta_+$, we have

$$\zeta_{L^p}(s) = -4\cdot 4^s \cdot (p+1)\,\zeta(k+\delta_+,\delta_1,\delta_2;s),$$

and therefore we can compute as follows:

$$\zeta'_{L^p}(0) = -4\log 4\cdot(p+1)\,\zeta(k+\delta_+,\delta_1,\delta_2;0) - 4(p+1)\,\zeta'(k+\delta_+,\delta_1,\delta_2;0).$$

By the results of [3], we have

$$\begin{aligned}\zeta(k+\delta_+,\delta_1,\delta_2;0) &= \zeta(-1,\delta_+) + \frac{(\delta_-)^2}{2} = -\frac{p+1}{4} - \frac{1}{12}, \\ \zeta'(k+\delta_+,\delta_1,\delta_2;0) &= \sum_{p=0}^{1}(\delta_-)^{1-p}(\zeta'(-p,\delta_1) - (-1)^p\zeta'(-p,\delta_2)) - (\delta_-)^2 \\ &= 2\zeta'(-1) + \sum_{k=2}^{p+1} k\log k - \delta_-\sum_{k=2}^{p+1}\log k - \left(\frac{p+1}{2}\right)^2,\end{aligned}$$

where $\zeta(s,a)$ is the Hurwicz zeta function defined by

$$\zeta(s,a) = \sum_{k=0}^{\infty}\frac{1}{(k+a)^s},$$

and $\zeta(s) = \zeta(s,1)$ is the Riemann zeta function.

Theorem 12. *For the line bundle L^p on the quadric Q^2, the analytic torsion* A-Tor(L^p) *is given by*

$$\text{A-Tor}(L^p) = -8(p+1)\,\zeta'(-1) + \log 4 \cdot (p+1)\left((p+1) + \frac{1}{3}\right) - 4(p+1)\sum_{k=1}^{p+1} k\log k + 2(p+1)^2\log((p+1)!) + (p+1)^3.$$

For the case $n = 3$, we have from Theorem 11

$$\begin{aligned}\zeta_{L^p}(s) = &-\frac{(12)^s}{3}\left(\zeta\left(\left(k+\frac{2p+7}{2}\right)^3, 2, 2p+5; s\right) - \frac{1}{4}\zeta\left(k+\frac{2p+7}{2}, 2, 2p+5; s\right)\right)\\ &-\frac{4(p+1)}{3}\cdot 6^s\left(\zeta\left(\left(k+\frac{p+4}{2}\right)^3, 1, p+3; s\right) - \frac{(p+1)^2}{4}\zeta\left(k+\frac{p+4}{2}, 1, p+3; s\right)\right)\\ &+\frac{4(p+2)}{3}\cdot 6^s\left(\zeta\left(\left(k+\frac{p+5}{2}\right)^3, 2, p+3; s\right) - \frac{(p+2)^2}{4}\zeta\left(k+\frac{p+5}{2}, 2, p+3; s\right)\right).\end{aligned}$$

Using the results of [3], we have

$$\zeta\left((k+\delta_+)^3, \delta_1, \delta_2; 0\right) = \zeta(-3, \delta_+) + \frac{1}{4}(\delta_-)^4,$$

$$\zeta'\left((k+\delta_+)^3, \delta_1, \delta_2; 0\right) = \sum_{p=0}^{3}\binom{3}{p}(\delta_-)^{3-p}(\zeta'(-p,\delta_1) - (-1)^p\zeta'(-p,\delta_2)) - \frac{2}{3}(\delta_-)^4.$$

Theorem 13. *For the line bundle L^p on the quadric Q^3, the analytic torsion* A-Tor(L^p) *is given by*

$$\begin{aligned}&\text{A-Tor}(L^p)\\ &\quad= 2\zeta'(-3) - (6p^2+18p+13)\,\zeta'(-1)\\ &\quad\quad -\frac{1}{3}\sum_{k=2}^{2p+4} k^3\log k + \frac{2p+3}{2}\sum_{k=2}^{2p+4} k^2\log k - \left(p^2+3p+\frac{13}{6}\right)\sum_{k=2}^{2p+4} k\log k\\ &\quad\quad +\frac{1}{6}(p+1)(p+2)(2p+3)\sum_{k=2}^{2p+4}\log k\end{aligned}$$

(continues)

$$+ \frac{4}{3}\sum_{k=2}^{p+2} k^3 \log k - \left(2p^2 + 6p + \frac{13}{3}\right)\sum_{k=2}^{p+2} k \log k$$
$$+ \frac{1}{3}(p+1)(p+2)(2p+3)\sum_{k=2}^{p+2} \log k$$
$$+ \frac{17}{36}p^4 + \frac{17}{6}p^3 + \frac{56}{9}p^2 + \frac{71}{12}p + \frac{295}{144}$$
$$+ \left(\frac{1}{2}p^3 + \frac{10}{3}p^2 + \frac{15}{2}p + \frac{511}{90}\right)\log 12$$
$$- \left(\frac{5}{6}p^2 + \frac{7}{2}p + \frac{653}{180}\right)\log 6.$$

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Matsugasaki Goshokaido-cho, Sakyo-ku, Kyoto, 606-8585, Japan.
E-mail address: `chiaki@kit.ac.jp`

要約

複素二次曲面上の線束の解析的ねじれを、線束のドルボー複体のラプラシアンのスペクトルから定まるスペクトル・ゼータ関数から、定義の通りに直接計算することを実行した。必要なスペクトル・データは、著者の [2] において求められた、対称対 (SO(n+2), SO(2)×SO(n)) についての表現の分岐則を用いて計算されている。求められたスペクトル・ゼータ関数は著者の [3] において考察されたものと同様の形をしていて、その原点での微分は、[3] の定理を用いて計算される。この結果は良く知られた Kai Köhler の論文 [1] におけるものと一致している。スペクトル・ゼータ関数の足し合わせにおける打消し合いを、スペクトル・データの所で観察することとなる。

キーワード： 解析的ねじれ、表現の分岐則、スペクトル・ゼータ関数。