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Conformal mapping between rectangles with a crossing slit

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Abstract

We study conformal mapping between two doubly connected domains— rectangles with a crossing slit. The ordered set of four vertices and the slit of the domain shall respectively correspond to those of the image domain. We will find the image rectangle and determine the modulus and the size of the crossing slit.

Key words: *conformal mapping, doubly connected domain, slit, modulus.*

1 Introduction

It is interesting to construct a conformal mapping between given domains. The first author gave in [2] a conformal mapping from a longitudinally horizontal slit rectangle onto a rectangle with a square removed. In this paper we will give a conformal mapping from the same domain to a rectangle with a crossing slit. These investigations are positioned as the generalization of the work in [6]. Let $\tau > 0$ and consider in the complex plane a rectangle $R(\tau) = Q_1Q_2Q_3Q_4$ with the modulus τ, where the coordinates of Q_1, Q_2, Q_3, Q_4 are $-1 + i\tau, -1 - i\tau, 1 - i\tau, 1 + i\tau$ respectively. Take $0 \leq \xi < 1$, $0 \leq \eta < \tau$ and set $S(\xi, \eta) = [-\xi, \xi] \bigcup [-i\eta, i\eta]$. We call $S(\xi, \eta)$ a crossing slit. A vertical slit $S(0, \eta)$ and a horizontal slit $S(\xi, 0)$ will be also regarded as crossing slits. We call the doubly connected domain $R(\tau, \xi, \eta) = R(\tau) - S(\xi, \eta)$ a rectangle with a crossing slit $S(\xi, \eta)$. A conformal mapping between rectangles with a crossing slit is always assumed to map the ordered set of vertices of rectangle to that of image rectangle in a natural manner.

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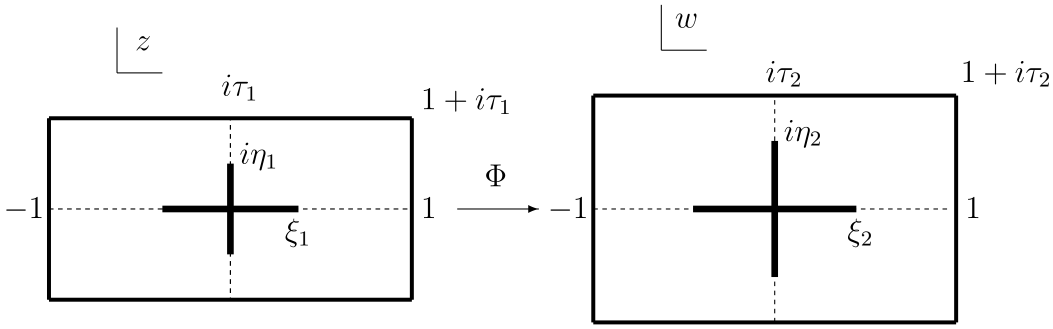

Figure 1: Conformal mapping between slit rectangles

We are interested in finding conditions under which there exists a conformal mapping between two rectangles with a crossing slit $R(\tau_1, \xi_1, \eta_1)$ and $R(\tau_2, \xi_2, \eta_2)$. Specifically we study the quantitative relation between τ_1, ξ_1, η_1 and τ_2, ξ_2, η_2. To this end it suffices to consider a special case where $\eta_1 = 0$.

Now, our problem reduces to the problem to study conformal mapping

$$\Phi : R(\tau_0, \xi_0, 0) \to R(\tau, \xi, \eta).$$

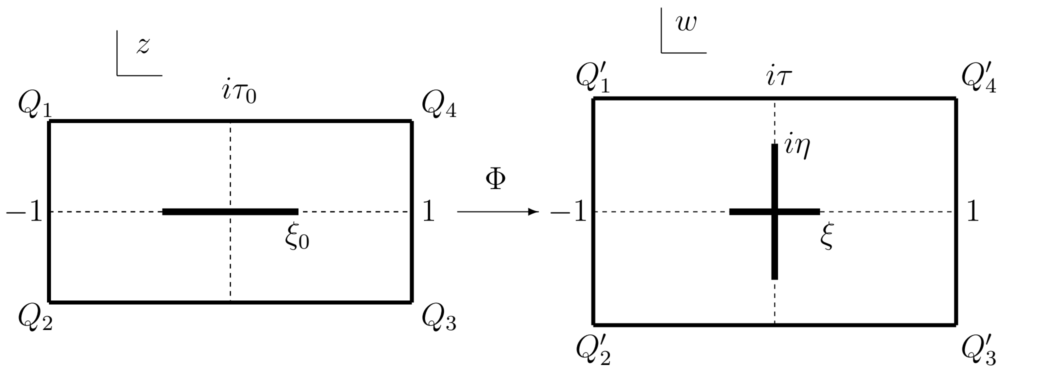

Figure 2: Reduction of our problem to a simpler case

2 Uniqueness

We begin with

Theorem 1. *Let* $\Phi_i (i = 1, 2)$ *be conformal mappings of* $R(\tau_1, \xi_1, \eta_1)$ *onto* $R(\tau_2, \xi_2, \eta_2)$. *Then* $\Phi_1 = \Phi_2$.

Proof. Take an annulus $A = \{z : \rho < |z| < 1\}$ which is conformally equivalent to $R(\tau_1, \xi_1, \eta_1)$ and let p be a conformal mapping of A onto $R(\tau_1, \xi_1, \eta_1)$. The composition $f = p^{-1} \circ \Phi_2^{-1} \circ \Phi_1 \circ p$ is a conformal self mapping of A which fixes a boundary point $p^{-1}(Q_1)$.

$$\begin{tikzcd} A \arrow[d, "p"'] & \\ R(\tau_1, \xi_1, \eta_1) \arrow[r, bend left, "\Phi_1"] & R(\tau_2, \xi_2, \eta_2) \arrow[l, bend left, "\Phi_2"] \end{tikzcd}$$

By the reflection principle (see [4]), we can extend f to a conformal mapping $\tilde{f}$ of the twice punctured sphere $\hat{\mathbb{C}} \setminus \{0, \infty\}$. The origin and the point at infinity are removable singularities of $\tilde{f}$ and hence $\tilde{f}$ is a Möbius transformation with three fixed points. Therefore $\tilde{f}$ is the identity and $\Phi_1 = \Phi_2$.

Corollary 1. *Let* Φ *be a conformal mapping of* $R(\tau_1, \xi_1, \eta_1)$ *to* $R(\tau_2, \xi_2, \eta_2)$. *Then* $\Phi(z) = \overline{\Phi(\bar{z})} = -\Phi(-z)$.

This corollary shows that for the construction of the conformal mapping $\Phi : R(\tau_0, \xi_0, 0) \to R(\tau, \xi, \eta)$, it suffices to study Φ only in the first quadrangle.

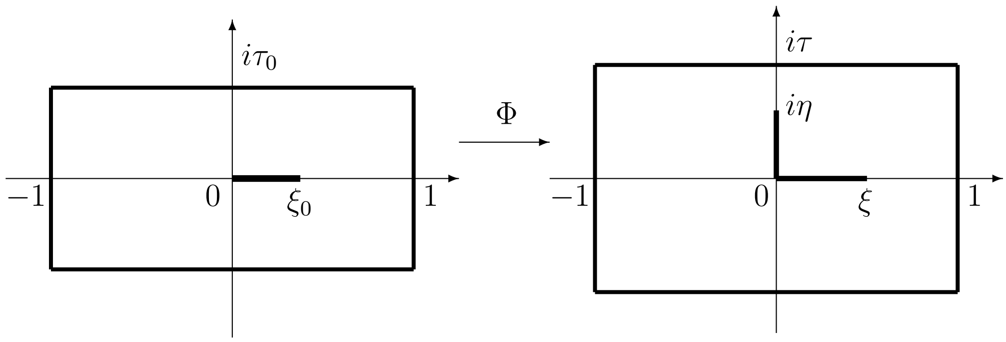

Figure 3: Decomposition of the rectangle into four symmetric rectangles

3 Construction of the conformal mapping

We next prove

Theorem 2. *For* $0 \leq \xi \leq \xi_0$, *there uniquely exist positive numbers* τ *and* η *and a conformal mapping* Φ_ξ *of* $R(\tau_0, \xi_0, 0)$ *to* $R(\tau, \xi, \eta)$.

Proof. Below the upper half plane is denoted by **H**. Let $\tilde{R}(\tau)$ and $\tilde{R}(\tau, \xi, \eta)$ denote the parts of $R(\tau)$ and $R(\tau, \xi, \eta)$ in the first quadrangle. For our purpose, it suffices to construct Φ_ξ from $\tilde{R}(\tau_0, \xi_0, 0)$ to $\tilde{R}(\tau, \xi, \eta)$. There exists a unique k $(0 < k < 1)$ and a unique conformal mapping $\psi : \tilde{R}(\tau_0) \to \mathbf{H}$ which maps the vertices $i\tau_0,\ 0,\ 1,\ 1 + i\tau_0$ to $-1/k,\ -1,\ 1,\ 1/k$ respectively. We set $s = \psi(\xi_0) \in (-1, 1)$. For t $(-1 \le t \le s)$ there exists a unique rectangle $\tilde{R}(\tau(t))$ $(\tau(t) > 0)$ and a unique conformal mapping $\varphi_t : \mathbf{H} \to \tilde{R}(\tau(t))$ which maps the points $-1/k,\ t,\ 1,\ 1/k$ on the real axis to the vertices $i\tau(t),\ 0,\ 1,\ 1 + i\tau(t)$. Note that $\tau(t)$ is uniquely determined for t by the conformal invariance of modulus and that $\tau(-1) = \tau_0$ and $\varphi_{-1} = \psi^{-1}$. Set $\eta(t) = \varphi_t(-1),\ \xi(t) = \varphi_t(s)$. We can represent φ_t by Schwarz-Christoffel formula (cf. [4]):

$$\varphi_t(w) = C_t \int_t^w \frac{dw}{\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}}, \tag{1}$$

where C_t is the real number satisfying:

$$C_t^{-1} = \int_t^1 \frac{dw}{\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}}. \tag{2}$$

The numbers $\tau(t), \xi(t), \eta(t)$ are given by the following:

$$\begin{aligned}
\tau(t) &= -iC_t \int_t^{-1/k} \frac{dw}{\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}}, && (3)\\
\xi(t) &= C_t \int_t^{s} \frac{dw}{\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}}, && (4)\\
\eta(t) &= -iC_t \int_t^{-1} \frac{dw}{\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}}. && (5)
\end{aligned}$$

The domain $\tilde{R}(\tau_0, \xi_0, 0)$ is the image of φ_{-1} and $\Phi_{\xi(t)} = \varphi_t \circ (\varphi_{-1})^{-1}$ gives the conformal mapping of $\tilde{R}(\tau_0, \xi_0, 0)$ onto $\tilde{R}(\tau(t), \xi(t), \eta(t))$. The $\xi(t)$ is clearly continuous and satisfies $\lim_{t \to -1} \xi(t) = 0,\ \lim_{t \to s} \xi(t) = \xi_0$. Therefore for a given $0 \le \xi \le \xi_0$ there exists a t which satisfies $\xi(t) = \xi$. The τ and η are given as $\tau(t)$ and $\eta(t)$. The modulus $\tau(t)$ is also continuous and monotonous by the argument of extremal length. As for C_t note that

$$1 = \varphi_t\left(\frac{1}{k}\right) - \varphi_t\left(-\frac{1}{k}\right) = C_t \left\{ \int_{-\frac{1}{k}}^{-\infty} + \int_{\infty}^{\frac{1}{k}} \right\} \frac{-dw}{\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}}$$

$$= C_t \int_{\frac{1}{k}}^{\infty} \left(\frac{1}{\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}} + \frac{1}{\sqrt{(w^2 - 1/k^2)(w + t)(w + 1)}} \right) dw.$$

And it follows that

$$0 = \frac{dC_t}{dt} \int_{\frac{1}{k}}^{\infty} \left(\frac{1}{\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}} + \frac{1}{\sqrt{(w^2 - 1/k^2)(w + t)(w + 1)}} \right) dw$$

$$+ \frac{C_t}{2} \int_{\frac{1}{k}}^{\infty} \left(\frac{1}{(w - t)\sqrt{(w^2 - 1/k^2)(w - t)(w - 1)}} - \frac{1}{(w + t)\sqrt{(w^2 - 1/k^2)(w + t)(w + 1)}} \right) dw$$

$$= \frac{dC_t}{dt} A(t) + \frac{C_t}{2} B(t),$$

where $A(t)$ and $B(t)$ denote the integral parts of the first and second term respectively. We get $\frac{dC_t}{dt} = -\frac{C_t B(t)}{2A(t)} < 0$. Hence C_t is decreasing. By

$$1 - \xi(t) = C_t \int_s^1 \frac{dw}{\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}},$$

we have

$$\frac{d\xi(t)}{dt} = -\frac{dC_t}{dt} \int_s^1 \frac{dw}{\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}}$$
$$-\frac{C_t}{2} \int_s^1 \frac{dw}{(w-t)\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}},$$

$$\frac{2A(t)}{C_t} \frac{d\xi(t)}{dt} = B(t) \int_s^1 \frac{dw}{\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}}$$
$$-A(t) \int_s^1 \frac{dw}{(w-t)\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}}$$
$$< \int_{\frac{1}{k}}^{\infty} \frac{dw}{(w-t)\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}} \int_s^1 \frac{dw}{\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}}$$
$$- \int_{\frac{1}{k}}^{\infty} \frac{dw}{\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}} \int_s^1 \frac{dw}{(w-t)\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}}$$
$$< \left(\frac{1}{\frac{1}{k} - t} - \frac{1}{1-t} \right) \int_{\frac{1}{k}}^{\infty} \frac{dw}{\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}} \int_s^1 \frac{dw}{\sqrt{(w^2 - 1/k^2)(w-t)(w-1)}}$$

< 0. Since $\frac{2A(t)}{C_t}$ is positive, it follows that $\frac{d\xi(t)}{dt}$ is negative and $\xi(t)$ is decreasing. Thus for given ξ $(0 \le \xi \le \xi_0)$ the equation $\xi(t) = \xi$ has a unique solution t $(-1 \le t \le s)$.

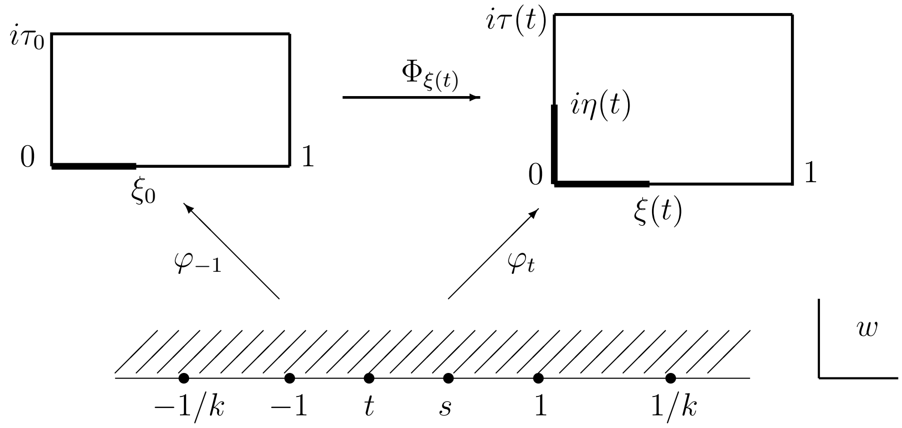

Figure 4: Construction of Φ_ξ via the upper half plane

We constructed the conformal mapping Φ by use of elliptic integrals. For other information and applications of Schwarz-Christoffel formula and elliptic integrals, see [1], [3], [4] and [5].

4 Examples

Some numerical examples of the modulus τ and the size ξ, η of the crossing slit are given in the next table. When the length of the horizontal slit decreases, the length of the vertical slit increases while the modulus τ is always increasing.

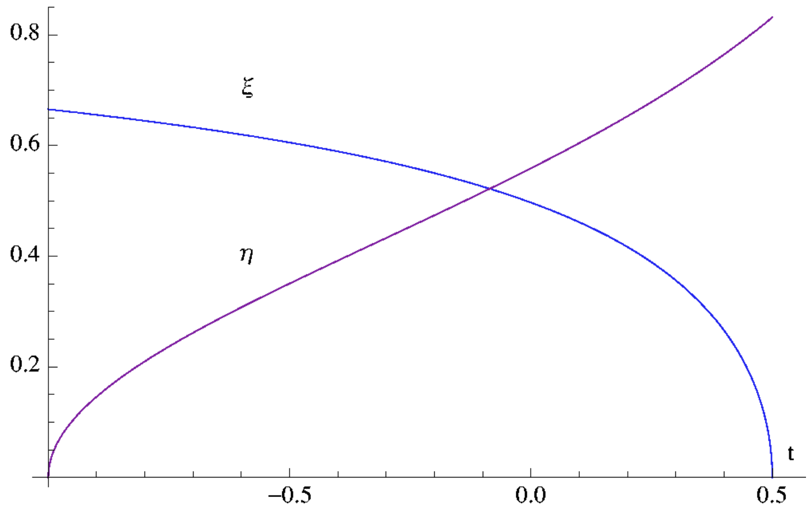

Figure 5: Crossing slit for $k = 0.2$, $s = 0.5$

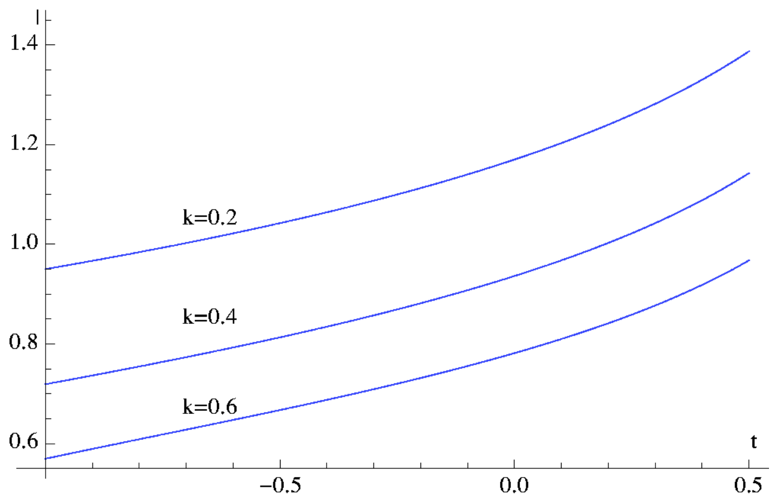

Figure 6: Modulus τ for $s = 0.5$

Table 1: Moduli

k	τ_0	s	ξ_0	t	$\tau(t)$	$\xi(t)$	$\eta(t)$
0.2	0.950	-0.5	0.335	-1.000	0.950	0.335	0.000
				-0.875	0.971	0.296	0.164
				-0.750	0.993	0.247	0.237
				-0.625	1.017	0.179	0.296
				-0.500	1.043	0.000	0.350
		0.0	0.500	-1.000	0.950	0.500	0.000
				-0.750	0.993	0.453	0.237
				-0.500	1.043	0.390	0.350
				-0.250	1.100	0.293	0.453
				0.000	1.170	0.000	0.559
		0.5	0.665	-1.000	0.950	0.665	0.000
				-0.625	1.017	0.623	0.296
				-0.250	1.100	0.561	0.453
				0.125	1.212	0.451	0.616
				0.500	1.387	0.000	0.831
0.4	0.719	-0.5	0.339	-1.000	0.719	0.339	0.000
				-0.875	0.741	0.298	0.168
				-0.750	0.764	0.248	0.241
				-0.625	0.788	0.178	0.300
				-0.500	0.813	0.000	0.353
		0.0	0.500	-1.000	0.719	0.500	0.000
				-0.750	0.764	0.448	0.241
				-0.500	0.813	0.384	0.353
				-0.250	0.869	0.287	0.451
				0.000	0.936	0.000	0.551
		0.5	0.661	-1.000	0.719	0.661	0.000
				-0.625	0.788	0.615	0.300
				-0.250	0.869	0.550	0.451
				0.125	0.976	0.441	0.605
				0.500	1.143	0.000	0.809
0.6	0.570	-0.5	0.348	-1.000	0.570	0.348	0.000
				-0.875	0.594	0.302	0.177
				-0.750	0.619	0.248	0.250
				-0.625	0.643	0.177	0.307
				-0.500	0.667	0.000	0.357
		0.0	0.500	-1.00	0.570	0.500	0.000
				-0.750	0.619	0.442	0.250
				-0.500	0.667	0.373	0.357
				-0.250	0.720	0.276	0.447
				0.000	0.782	0.000	0.536
		0.5	0.652	-1.000	0.570	0.652	0.000
				-0.625	0.643	0.597	0.307
				-0.250	0.720	0.529	0.447
				0.125	0.818	0.422	0.584
				0.500	0.968	0.000	0.766

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