Memoirs of the Faculty of Engineering and Design
Kyoto Institute of Technology 49 (February, 2001)

Ahlfors-Rauch Type Variational Formulas on Complex Manifolds

Dedicated to Professor Hiroshi Yamaguchi on his sixtieth birthday

Fumio MAITANI

(Received August 29, 2000; Accepted September 11, 2000)

Abstract

We consider variational calculus of quantities, which are represented by the inner product of some specific harmonic forms with boundary behavior, under a certain small deformation of a complex manifold.

Key Words: *Complex manifold, Harmonic forms, Deformation, Variational formulas, Distortion theorem.*

1. Introduction

In this paper, we attempt to extend Ahlfors-Rauch type variational formulas [2), 7), 8)] on Riemann surfaces to those on complex manifolds. We have already given various variational formulas for potential theoretical quantities under quasiconformal deformation of a Riemann surface.[5), 6)] They have the same form as those of Ahlfors-Rauch. Inheriting the study, we consider a variational calculus of quantity, which is represented by the inner product of some specific harmonic forms with boundary behavior, under a certain small deformation of a complex manifold. For the description of small deformation we require a corresponding one to a quasiconformal mapping. Quasiconformal mappings are also discussed in real high dimensional Euclidean space, but they are treated in here from a different point of view. We introduce Beltrami tensors of diffeomorphisms which act as Beltrami coefficients of quasiconformal mappings. The Beltrami tensor, as it were, indicates the degree of its distortion. For the definition, we use Hodge's conjugate operators on complex manifolds with hermitian metrics. When a diffeomorphism is interposed between Hodge's conjugate operators, the reflexive property of Hodge's conjugate operator is lost. As for harmonic forms with boundary behavior, we use a certain Hilbert space devised from the Hilbert space of square integrable forms. This allows us to use the same methods as those used with Riemann surfaces. Under these circumstances, the degree of distortion of a certain harmonic form with boundary behavior is estimated by the norm of the Beltrami tensor. Making use of this estimation, we can obtain our Ahlfors-Rauch type variational formulas.

The present study is supported by Grant-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan, #1264 0171

2. Forms and the pull back by a diffeomorphism

Let M (resp. $\tilde{M}$) be an n-dimensional complex manifold with a hermitian metric $g_{\alpha\bar{\beta}}dz^{\alpha}\otimes d\bar{z}^{\beta}$ (resp. $\tilde{g}_{\alpha\bar{\beta}}dw^{\alpha}\otimes d\bar{w}^{\beta}$), where $g_{\alpha\bar{\beta}}(z)$ (resp. $\tilde{g}_{\alpha\bar{\beta}}(w)$) belongs to the C^{∞} class. Assume that there is an oriented C^{∞} homeomorphism f from M to $\tilde{M}$. We shall consider differential forms and tensors, then make use of Einstein's summation convention, with which we agree to sum over the possible values (represented if necessary) of the indices with respect to each index which appears twice, once at the top and once at the bottom. We shall also use the Kronecker symbol $\delta_{i_1\dots i_p}^{j_1\dots j_p}$, which is zero unless $i_1\dots i_p$ and $j_1\dots j_p$ are derangements of the same p distinct integers, and which is $+1$ when $i_1\dots i_p$ is an even derangement of $j_1\dots j_p$ and is -1 when it is an odd derangement. Note that $\delta_{i_1\dots i_p}^{j_1\dots j_p}$ is a tensor. Now a (p,q) form φ on $\tilde{M}$ can be written in the form:

$$\varphi=\varphi^{(p,q)}=\frac{1}{p!q!}\varphi_{\alpha_1\dots\alpha_p\bar{\beta}_1\dots\bar{\beta}_q}dw^{\alpha_1}\wedge\dots\wedge dw^{\alpha_p}\wedge d\bar{w}^{\beta_1}\wedge\dots\wedge d\bar{w}^{\beta_q}$$

$(1\le\alpha_i\le n, 1\le\beta_i\le n)$, where $\varphi_{\alpha_1\dots\alpha_p\bar{\beta}_1\dots\bar{\beta}_q}$ is skew symmetric with respect to $\alpha_1,\dots,\alpha_p$ and also $\beta_1,\dots,\beta_q$. Abbreviating, we denote this by

$$\varphi=\varphi_{A_p\bar{B}_q}dw^{A_p}\wedge d\bar{w}^{B_q}, \tag{2.1}$$

where the summation is taken over $A_p=\alpha_1\dots\alpha_p;\alpha_1<\alpha_2\dots<\alpha_p$, $\beta_q=\beta_1\dots\beta_q;\beta_1<\beta_2<\dots<\beta_q$. Now the pull back of φ by f is

$$\varphi\circ f=\sum_{s+t=p+q}\frac{1}{p!q!s!t!}\varphi_{\alpha_1\dots\alpha_p\bar{\beta}_1\dots\bar{\beta}_q}(f(z))\frac{\partial(w^{\alpha_1}\dots w^{\alpha_p}\bar{w}^{\beta_1}\dots\bar{w}^{\beta_q})}{\partial(z^{\kappa_1}\dots z^{\kappa_s}\bar{z}^{\lambda_1}\dots\bar{z}^{\lambda_t})}$$
$$\times dz^{\kappa_1}\wedge\dots\wedge dz^{\kappa_s}\wedge d\bar{z}^{\lambda_1}\wedge\dots\wedge d\bar{z}^{\lambda_t}.$$

We denote the Jacobian by $\dfrac{\partial(w^{A_p}\bar{w}^{B_q})}{\partial(z^{K_s}\bar{z}^{\Lambda_t})}$ or $J_{K_s\bar{\Lambda}_t}^{A_p\bar{B}_q}$ and write the pull back as

$$\varphi\circ f=\sum_{s+t=p+q}\varphi_{A_p\bar{B}_q}J_{K_s\bar{\Lambda}_t}^{A_p\bar{B}_q}dz^{K_s}\wedge d\bar{z}^{\Lambda_t}. \tag{2.2}$$

The inverse matrix of $(\tilde{g}_{\alpha\bar{\beta}}(w))_{\alpha,\beta=1,\dots,n}$ is denoted by $(\tilde{g}^{\bar{\alpha}\beta}(w))$ and $\{\tilde{g}^{\bar{\alpha}\beta}(w)\}$ are skew symmetric contravariant tensor fields. They satisfy $\tilde{g}_{\alpha\bar{r}}\tilde{g}^{\bar{r}\beta}=\tilde{g}^{\bar{\beta}r}\tilde{g}_{r\bar{\alpha}}=\delta_{\alpha}^{\beta}$. Let $\tilde{g}(w)$ be the skew symmetric covariant tensor field $\det(\tilde{g}_{\alpha\bar{\beta}}(w))$ of rank $2n$. Set

$$\tilde{g}_{B_qB_{n-q}\bar{A}_p\bar{A}_{n-p}}(w)=\delta_{\beta_1\dots\beta_q\beta_{q+1}\dots\beta_n}^{\alpha_1\dots\alpha_p\alpha_{p+1}\dots\alpha_n}\tilde{g}(w), \tag{2.3}$$

where

$$A_{n-p}=\alpha_{p+1}\dots\alpha_n;\alpha_{p+1}<\alpha_{p+2}<\dots<\alpha_n,\ \{\alpha_{p+1},\dots,\alpha_n\}\cap\{\alpha_1,\dots,\alpha_p\}=\emptyset,$$

$$B_{n-q}=\beta_{q+1}\dots\beta_n;\beta_{q+1}<\beta_{q+2}<\dots<\beta_n,\ \{\beta_{q+1},\dots,\beta_n\}\cap\{\beta_1,\dots,\beta_q\}=\emptyset.$$

Set

$$\tilde{g}^{\bar{A}_p H_p}\tilde{g}^{\bar{\Xi}_q B_q}=\tilde{g}^{\bar{\alpha}_1\eta_1}\ldots\tilde{g}^{\bar{\alpha}_p\eta_p}\tilde{g}^{\bar{\xi}_1\beta_1}\ldots\tilde{g}^{\bar{\xi}_q\beta_q} \tag{2.4}$$

and

$$\varphi^{\bar{A}_p\beta_q}(w)=\tilde{g}^{\bar{A}_p H_p}\tilde{g}^{\bar{\Xi}_q B_q}\varphi_{H_p\bar{\Xi}_q}(w). \tag{2.5}$$

Then, Hodge's conjugate operator $*$ is defined by

$$*\varphi=i^n(-1)^{\frac{n(n-1)}{2}+np}\tilde{g}_{B_q B_{n-q}\bar{A}_p\bar{A}_{n-p}}\varphi^{\bar{A}_p B_q}(w)\,dw^{B_{n-q}}\wedge d\bar{w}^{A_{n-p}}. \tag{2.6}$$

This operator $*$ transforms a (p,q) form φ to an $(n-q,n-p)$ form $*\varphi$ which is called the dual form of φ. This satisfies

$$*(c_1\varphi_1+c_2\varphi_2)=c_1*\varphi_1+c_2*\varphi_2,$$

$$\overline{(*\varphi)}=*(\bar{\varphi}),\ (\ \bar{}\ \text{ denotes the complex conjugate}),$$

$$*(*\varphi)=(-1)^{p+q}\varphi.$$

The pull back of $*\varphi$ by f is

$$(*\varphi)\circ f=\sum_{s+t=p+q}i^n(-1)^{\frac{n(n-1)}{2}+np}\tilde{g}_{B_q B_{n-q}\bar{A}_p\bar{A}_{n-p}}\tilde{g}^{\bar{A}_p H_p}\tilde{g}^{\bar{\Xi}_q B_q}\varphi_{H_p\bar{\Xi}_q}$$
$$\times\frac{\partial(w^{B_{n-q}}\bar{w}^{A_{n-p}})}{\partial(z^{\Lambda_{n-t}}\bar{z}^{K_{n-s}})}dz^{\Lambda_{n-t}}\wedge d\bar{z}^{K_{n-s}}.$$

We have

$$*((*\varphi)\circ f)=\sum_{s+t=p+q}i^n(-1)^{\frac{n(n-1)}{2}+n(n-t)}g_{K_{n-s}K_s\bar{\Lambda}_{n-t}\bar{\Lambda}_t}g^{\bar{\Lambda}_{n-t}\Gamma_{n-t}}g^{\bar{E}_{n-s}K_{n-s}}$$
$$\times i^n(-1)^{\frac{n(n-1)}{2}+np}\tilde{g}_{B_q B_{n-q}\bar{A}_p\bar{A}_{n-p}}\tilde{g}^{\bar{A}_p H_p}\tilde{g}^{\bar{\Xi}_q B_q}\varphi_{H_p\bar{\Xi}_q}(f(z))$$
$$\times\frac{\partial(w^{B_{n-q}}\bar{w}^{A_{n-p}})}{\partial(z^{\Gamma_{n-t}}\bar{z}^{E_{n-s}})}dz^{K_s}\wedge d\bar{z}^{\Lambda_t}.$$

Note that $g_{K_{n-s}K_s\bar{\Lambda}_{n-t}\bar{\Lambda}_t}=(-1)^{(n-s)s+(n-t)t}g_{K_sK_{n-s}\bar{\Lambda}_t\bar{\Lambda}_{n-t}}$. Changing the indices $A(B)$ and $H(\Xi)$, set

$$L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}=(-1)^{n(p+s)}g_{K_sK_{n-s}\bar{\Lambda}_t\bar{\Lambda}_{n-t}}g^{\bar{\Lambda}_{n-t}\Gamma_{n-t}}g^{\bar{E}_{n-s}K_{n-s}}$$
$$\times\tilde{g}_{\Xi_q\Xi_{n-q}\bar{H}_p\bar{H}_{n-p}}\tilde{g}^{\bar{H}_pA_p}\tilde{g}^{\bar{B}_q\Xi_q}\frac{\partial(w^{\Xi_{n-q}}\bar{w}^{H_{n-p}})}{\partial(z^{\Gamma_{n-t}}\bar{z}^{E_{n-s}})}. \tag{2.7}$$

We can obtain

$$(-1)^{p+q}*((*\varphi)\circ f)=\sum_{s+t=p+q}L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}\varphi_{A_p\bar{B}_q}(f(z))\,dz^{K_s}\wedge d\bar{z}^{\Lambda_t} \tag{2.8}$$

and

$$\varphi\circ f-(-1)^{p+q}*((*\varphi)\circ f)=\sum_{s+t=p+q}\varphi_{A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}-L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t})dz^{K_s}\wedge d\bar{z}^{\Lambda_t}. \tag{2.9}$$

Let $\tilde{\omega}=i\tilde{g}_{\alpha\bar{\beta}}dw^{\alpha}\wedge d\bar{w}^{\beta}$. Then

$$\begin{aligned}\frac{\tilde{\omega}^n}{n!} &= \frac{i^n}{n!}\tilde{g}_{\alpha_1\bar{\beta}_1}\ldots\tilde{g}_{\alpha_n\bar{\beta}_n}dw^{\alpha_1}\wedge d\bar{w}^{\beta_1}\wedge\ldots\wedge dw^{\alpha_n}\wedge d\bar{w}^{\beta_n}\\ &=2^n\det(\tilde{g}_{\alpha\bar{\beta}})dx^1\wedge\ldots\wedge dx^{2n}\qquad(w^{\alpha}=x^{2\alpha-1}+ix^{2\alpha}),\end{aligned} \tag{2.10}$$

which is called the volume element. For two forms φ and $\psi, n!(\varphi\wedge*\bar{\psi})/\omega^n$ is a function on $\tilde{M}$ and the scalar product on $\tilde{M}$ is defined as follows

$$(\varphi,\psi)_{\tilde{M}}=\iint_{\tilde{M}}\varphi\wedge*\bar{\psi}=\iint_{\tilde{M}}\frac{\varphi\wedge*\bar{\psi}}{\tilde{\omega}^n}\tilde{\omega}^n \tag{2.11}$$

provided that the integral converges. If we choose a local variable w about w_0 so that $\tilde{g}_{\alpha\bar{\beta}}(w_0)=\delta^{\alpha}_{\beta}$, then $\tilde{g}(w_0)=1$ and $\psi^{A_p\bar{B}_q}(w_0)=\psi_{A_p\bar{B}_q}(w_0)$. For (p,q) forms φ and $\psi, \varphi\wedge*\bar{\psi}=\tilde{\omega}^n\varphi_{A_p\bar{B}_q}\overline{\psi^{A_p\bar{B}_q}}/n!$, hence $\varphi_{A_p\bar{B}_q}\overline{\psi^{A_p\bar{B}_q}}(w_0)$ is a function on $\tilde{M}$ for variable w_0. Further, we choose a local variable z about z_0 so that $\tilde{g}_{\alpha\bar{\beta}}(z_0)=\delta^{\alpha}_{\beta}(w_0=f(z_0))$. In these variables it follows that at z_0

$$L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}=(-1)^{n(p+s)+pq+st+\Sigma^p_{i=1}\alpha_i+\Sigma^q_{i=1}\beta_i+\Sigma^s_{i=1}\kappa_i+\Sigma^t_{i=1}\lambda_i}J^{B_{n-q}\bar{A}_{n-p}}_{\Lambda_{n-t}\bar{K}_{n-s}}.$$

We have

$$\overline{J^{B_q\bar{A}_p}_{\Lambda_t\bar{K}_s}}=(-1)^{pq+st}J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}\ \text{and}\ \overline{L^{B_q\bar{A}_p}_{\Lambda_t\bar{K}_s}}=(-1)^{pq+st}L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}. \tag{2.12}$$

By Laplace's expansion theorem

$$\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}}J^{H_u\bar{\Xi}_q}_{K_s\bar{\Lambda}_t}\overline{L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}}=\begin{cases}\dfrac{\partial(w^1\ldots w^n\bar{w}^1\ldots\bar{w}^n)}{\partial(z^1\ldots z^n\bar{z}^1\ldots\bar{z}^n)} & \text{if } H_u=A_p \text{ and } \Xi_v=B_q,\\ 0 & \text{the other case}\end{cases} \tag{2.13}$$

Hereafter, the Jacobian will be denoted by J. From eqs. (2.2), (2.6) and (2.8) we can get

$$(\varphi\circ f\wedge\overline{(*\psi)\circ f})\frac{n!}{\omega^n}=\sum_{A_p,B_q}\varphi_{A_p\bar{B}_q}\overline{\psi_{A_p\bar{B}_q}}J\ \ at\ z_0.$$

This allows us to conclude the following.

Lemma 2.1. For (p,q) forms φ and ψ

$$(\varphi\circ f,(-1)^{p+q}*((*\psi)\circ f))_M=(\varphi,\psi)_{\tilde{M}}. \tag{2.14}$$

3. Beltrami tensors

In this section, we define Beltrami tensors which correspond to Beltrami coefficients on a Riemann surface. For a (p,q) form φ we have the representations

$$\varphi\circ f-(-1)^{p+q}*((*\varphi)\circ f) = \sum_{s+t=p+q} \varphi_{A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}-L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t})dz^{K_s}\wedge d\bar{z}^{\Lambda_t},$$

$$\varphi\circ f+(-1)^{p+q}*((*\varphi)\circ f) = \sum_{s+t=p+q} \varphi_{A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}+L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t})dz^{K_s}\wedge d\bar{z}^{\Lambda_t}.$$

We consider simultaneous linear equations with $\sum_{u+v=s+t} {}_nC_u\,{}_nC_v$ (cf. ${}_nC_k$ is the notation of combination) unknowns $\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}$ which satisfy

$$J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}-L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}=\sum_{u+v=p+q}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}+L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t} \tag{3.1}$$

for every $(A_p,B_q)(p+q=s+t)$, where K_s and Λ_t are fixed. Assume that the solutions $\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}$ exist and are uniquely determined. Let $\mu^{\eta_1\ldots\eta_u\bar{\xi}_1\ldots\bar{\xi}_v}_{\kappa_1\ldots\kappa_s\bar{\lambda}_1\ldots\bar{\lambda}_t}$ denote one that is defined skew symmetrically with respect to $\eta_i,\xi_i,\kappa_i,\lambda_i$ from $\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}$. Then, we can verify $\mu^{\eta_1\ldots\eta_u\bar{\xi}_1\ldots\bar{\xi}_v}_{\kappa_1\ldots\kappa_s\bar{\lambda}_1\ldots\bar{\lambda}_t}$ becomes a tensor and call it a Beltrami tensor. This terminology is derived from $\partial w/\partial\bar{z}=(\partial w/\partial z)\mu^{\bar{1}}_{1}$ for $n=1$. Now

$$\mu^{\eta_1\ldots\eta_u\bar{\xi}_1\ldots\bar{\xi}_v}_{\kappa_1\ldots\kappa_s\bar{\lambda}_1\ldots\bar{\lambda}_t}\overline{\mu^{\alpha_1\ldots\alpha_u\bar{\beta}_1\ldots\bar{\beta}_v}_{\gamma_1\ldots\gamma_s\bar{\delta}_1\ldots\bar{\delta}_t}}\,g^{\bar{\gamma}_1\kappa_1}\ldots g^{\bar{\gamma}_s\kappa_s}g^{\bar{\lambda}_1\delta_1}\ldots g^{\bar{\lambda}_t\delta_t}g_{\eta_1\bar{\alpha}_1}\ldots g_{\eta_u\bar{\alpha}_u}g_{\beta_1\bar{\xi}_1}\ldots g_{\beta_v\bar{\xi}_v}$$

is independent of the choice of local variables, and hence is a function on M. If we choose the local variable z at z_0 so that $g_{\alpha\bar{\beta}}(z_0)=g^{\bar{\alpha}\beta}(z_0)=\delta^{\alpha}_{\beta}$, it has the value $\sum|\mu^{\eta_1\ldots\eta_u\bar{\xi}_1\ldots\bar{\xi}_v}_{\kappa_1\ldots\kappa_s\bar{\lambda}_1\ldots\bar{\lambda}_t}(z_0)|^2=u!v!s!t!\sum_{K_s,\Lambda_t}\sum_{H_u,\Xi_v}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}(z_0)|^2$ at z_0. Set

$$\mu_m(z_0)=\sum_{\substack{K_s\Lambda_t\\ s+t=m}}\sum_{\substack{K_s\Lambda_t\\ u+v=m}}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}(z_0)|^2 \text{ and } \|\mu_m\|=\sup_{z_0\in M}\mu_m(z_0).$$

We note some distortions.

Theorem 3.1. *Let* $\|\mu_m\|\le k^2<1$. *Then for a* (p,q) *form* φ $(\|\varphi\|<\infty)$ *on* $\tilde{M}$ $(p+q=m)$,

$$\|\varphi\circ f\|^2+\|(*\varphi)\circ f\|^2\le 2\,{}_nC_p\,{}_nC_q\frac{1+k^2}{1-k^2}\|\varphi\|^2, \tag{3.2}$$

$$\left\|\varphi\circ f-(-1)^{p+q}*((*\varphi)\circ f)\right\|\le k\left\|\varphi\circ f+(-1)^{p+q}*((*\varphi)\circ f)\right\|. \tag{3.3}$$

Proof. Let z and w be local variables at $z_0, w_0=f(z_0)$ so that $g_{\alpha\bar{\beta}}(z_0)=g_{\alpha\bar{\beta}}(w_0)=\delta^{\alpha}_{\beta}$. Consider the functions

$$(\varphi\circ f\wedge *\overline{(\varphi\circ f)})\frac{n!}{\omega^n} \text{ and } ((*\varphi)\circ f\wedge *\overline{((*\varphi)\circ f)})\frac{n!}{\omega^n}.$$

We have from eqs. (2.2), (2.8)

$$\begin{aligned}(\varphi\circ f\wedge *\overline{(\varphi\circ f)})\frac{n!}{\omega^n}(z_0) &= \sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} (\varphi_{A_p\bar{B}_q} J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t})(z_0)\overline{(\varphi_{A_p\bar{B}_q} J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t})(z_0)}\\ &\le \sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} \Big\{\sum_{A_p,B_q}|\varphi_{A_p\bar{B}_q}(z_0)|^2 \sum_{A_p,B_q}|J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}(z_0)|^2\Big\}\end{aligned}$$

and

$$((*\varphi)\circ f\wedge *\overline{((*\varphi)\circ f)})\frac{n!}{\omega^n}(z_0)\le \sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} \Big\{\sum_{A_p,B_q}|\varphi_{A_p\bar{B}_q}(z_0)|^2 \sum_{A_p,B_q}|L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t})(z_0)|^2\Big\}.$$

Set

$$\begin{aligned}X^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} &= |J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}+L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2+|J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}-L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2,\\ Y^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} &= |J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}+L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2-|J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}-L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2.\end{aligned}$$

From eq. (3.1)

$$|J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}-L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2\le \sum_{\substack{H_u,\Xi_v\\ u+v=p+q}} |J^{A_p\bar{B}_q}_{H_s\bar{\Xi}_v}+L^{A_p\bar{B}_q}_{H_s\bar{\Xi}_v}|^2 \sum_{\substack{H_u,\Xi_v\\ u+v=p+q}} |\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}|^2.$$

Hence

$$\begin{aligned}\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} X^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} &\le \sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} |J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}+L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2\Big\{1+\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}}\sum_{\substack{H_u\Xi_v\\ u+v=p+q}}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}|^2\Big\},\\ \sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} Y^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} &\ge \sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} |J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}+L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2\Big\{1-\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}}\sum_{\substack{H_u\Xi_v\\ u+v=p+q}}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}|^2\Big\}.\end{aligned}$$

From the normalization and eq. (2.13) $\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} Y^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} = 4J$. Therefore,

$$\begin{aligned}\frac{1}{J}\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}}\Big\{|J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2+|L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2\Big\} &= 2\frac{\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} X^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}}{\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}} Y^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}}\\ &\le 2\frac{1+\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}}\sum_{\substack{H_u\Xi_v\\ u+v=p+q}}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}|^2}{1-\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}}\sum_{\substack{H_u\Xi_v\\ u+v=p+q}}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}|^2}\le 2\frac{1+k^2}{1-k^2}\end{aligned}$$

It follows that

$$\|\varphi\circ f\|^2+\|(*\varphi)\circ f\|^2=\iint(\varphi\circ f\wedge *\overline{(\varphi\circ f)}+(*\varphi)\circ f\wedge *\overline{((*\varphi)\circ f)})$$
$$\le\iint\sum_{A_p B_q}|\varphi_{A_p\bar{B}_q}|^2 2\,{}_nC_p\,{}_nC_q\frac{1+\mu_m}{1-\mu_m}J\frac{\omega^n}{n!}$$
$$\le 2\,{}_nC_p\,{}_nC_q\frac{1+k^2}{1-k^2}\iint\sum_{A_p B_q}|\varphi_{A_p\bar{B}_q}|^2 J\frac{\omega^n}{n!}$$
$$=2\,{}_nC_p\,{}_nC_q\frac{1+k^2}{1-k^2}\|\varphi\|^2.$$

For ineq. (3.3) let $\Phi=\varphi\circ f-(-1)^{p+q}*((*\varphi)\circ f),\Psi=\varphi\circ f+(-1)^{p+q}*((*\varphi)\circ f)$ and write as eq. (2.9). We have

$$(\Phi\wedge *\overline{\Phi})\frac{n!}{\omega^n}(z_0)=\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}}|\varphi_{A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}+L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}(z_0)|^2$$
$$\le\sum_{\substack{K_s,\Lambda_t\\ s+t=p+q}}\Big\{\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}|\varphi_{A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}+L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})(z_0)|^2\Big\}\Big\{\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}(z_0)|^2\Big\}$$
$$=(\Psi\wedge *\overline{\Psi})\frac{n!}{\omega^n}(z_0)\mu_m(z_0).$$

Hence $\|\Phi\|\le k\|\Psi\|$.

We now consider Beltrami tensors of composed mappings. Let $M,\underline{\underline{M}}$ and $\tilde{M}$ be n-dimensional complex manifolds with hermitian metrics $g_{\alpha\bar{\beta}},\underline{\underline{g}}_{\alpha\bar{\beta}},\tilde{g}_{\alpha\bar{\beta}}$ and f (*resp.* g) be a diffeomorphism from M to $\underline{\underline{M}}$ (*resp.* from $\underline{\underline{M}}$ to $\tilde{M}$). Let $J,\underline{\underline{J}},\tilde{J}$ (*resp.* $L,\underline{\underline{L}},\tilde{L}$) denote those for f, g, $g\circ f$ as in eq. (2.2) (resp. eq. (2.7)) and $\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t},\underline{\underline{\mu}}^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t},\tilde{\mu}^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}$ be the Beltrami tensors of f, g, $g\circ f$, respectively. We choose local variables z, w, x at z_0, $w_0=f(z_0)$, $x_0=g(w_0)$ on M, $\underline{\underline{M}}$, $\tilde{M}$, respectively so that $g_{\alpha\bar{\beta}}(z_0)=\underline{\underline{g}}_{\alpha\bar{\beta}}(w_0)=\tilde{g}_{\alpha\bar{\beta}}(x_0)=\delta^{\beta}_{\alpha}$. Note that

$$\frac{\partial(x^{A_p}\bar{x}^{B_q})}{\partial(z^{H_u}\bar{z}^{\Xi_v})}=\sum_{\ell+m=p+q}\frac{\partial(x^{A_p}\bar{x}^{B_q})}{\partial(w^{\Gamma_\ell}\bar{w}^{E_m})}\frac{\partial(w^{\Gamma_\ell}\bar{w}^{E_m})}{\partial(z^{H_u}\bar{z}^{\Xi_v})}.$$

From this and eq. (2.13)

$$\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\tilde{J}^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}\overline{L^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v}}=\sum_{\ell+m=p+q}\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\underline{\underline{J}}^{A_p\bar{B}_q}_{\Gamma_\ell\bar{E}_m}J^{\Gamma_\ell\bar{E}_m}_{H_u\bar{\Xi}_v}\overline{L^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v}}=J\underline{\underline{J}}^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}.$$

Hence

$$\underline{\underline{J}}^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}=\frac{1}{J}\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\tilde{J}^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}\overline{L^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v}}.$$

Similarly

$$\underline{\underline{L}}^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}=\frac{1}{J}\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\tilde{L}^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}\overline{J^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v}}.$$

It thus follows that

$$
\begin{aligned}
J(J^{A_p\bar B_q}_{\underline{\underline{K_s\bar\Lambda_t}}}-L^{A_p\bar B_q}_{\underline{\underline{K_s\bar\Lambda_t}}}) &= \sum_{\substack{H_u,\Xi_v\\ u+v=p+q}} \left\{\tilde J^{A_p\bar B_q}_{H_u\bar\Xi_v}\overline{L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}-\tilde L^{A_p\bar B_q}_{H_u\bar\Xi_v}\overline{J^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}\right\}\\
&= \sum_{\substack{H_u,\Xi_v\\ u+v=p+q}} \left\{(\tilde J^{A_p\bar B_q}_{H_u\bar\Xi_v}-\tilde L^{A_p\bar B_q}_{H_u\bar\Xi_v})\overline{L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}-\overline{(J^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}-L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v})}\tilde L^{A_p\bar B_q}_{H_u\bar\Xi_v}\right\}\\
&= \sum_{\ell+m=p+q}\ \sum_{\substack{H_u,\Xi_v\\ u+v=p+q}} \left\{(\tilde J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}+\tilde L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m})\overline{L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}(\tilde\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v}-\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v})\right.\\
&\qquad\left.+(\tilde J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}-J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}+\tilde L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}-L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m})\overline{L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v}\right\}\\
&\quad-\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}} \left\{\overline{(J^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}-L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v})}(\tilde L^{A_p\bar B_q}_{H_u\bar\Xi_v}-L^{A_p\bar B_q}_{H_u\bar\Xi_v})-(J^{A_p\bar B_q}_{H_u\bar\Xi_v}\overline{L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}-\overline{J^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}L^{A_p\bar B_q}_{H_u\bar\Xi_v})\right\}.
\end{aligned}
$$

From eq. (2.13) the last term vanishes. On the other hand,

$$
\begin{aligned}
J(J^{A_p\bar B_q}_{\underline{\underline{K_s\bar\Lambda_t}}}-L^{A_p\bar B_q}_{\underline{\underline{K_s\bar\Lambda_t}}}) &= \sum_{u+v=p+q} J(J^{A_p\bar B_q}_{\underline{\underline{K_s\bar\Xi_t}}}+L^{A_p\bar B_q}_{\underline{\underline{K_s\bar\Xi_t}}})\underline{\underline{\mu}}^{H_u\bar\Xi_v}_{\underline{\underline{K_s\bar\Lambda_t}}}\\
&= \sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\ \sum_{\substack{\Gamma_\ell,E_m\\ \ell+m=p+q}} (\tilde J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}\overline{L^{H_u\bar\Xi_v}_{\Gamma_\ell\bar E_m}}+\tilde L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}\overline{J^{H_u\bar\Xi_v}_{\Gamma_\ell\bar E_m}})\underline{\underline{\mu}}^{H_u\bar\Xi_v}_{K_s\bar\Lambda_t}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\ \sum_{\substack{\Gamma_\ell,E_m\\ \ell+m=p+q}} (\tilde J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}\overline{L^{H_u\bar\Xi_v}_{\Gamma_\ell\bar E_m}}+\tilde L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}\overline{J^{H_u\bar\Xi_v}_{\Gamma_\ell\bar E_m}})\underline{\underline{\mu}}^{H_u\bar\Xi_v}_{K_s\bar\Lambda_t}\\
&= \sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\Big[\sum_{\ell+m=p+q}\left\{(\tilde J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}+\tilde L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m})\overline{L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}(\tilde\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v}-\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v})\right.\\
&\qquad\left.+(\tilde J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}-J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}+\tilde L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}-L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m})\overline{L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}}\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v}\right\}\\
&\qquad-\overline{(J^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v}-L^{K_s\bar\Lambda_t}_{H_u\bar\Xi_v})}(\tilde L^{A_p\bar B_q}_{H_u\bar\Xi_v}-L^{A_p\bar B_q}_{H_u\bar\Xi_v})\Big].
\end{aligned}
\tag{3.4}
$$

Now consider a family of n-dimensional complex manifolds M_τ with a parameter $\tau=(\tau_1,\tau_2,\dots,\tau_m)$ in a neighborhood of zero in m dimensional Euclidean space R^m and diffeomorphisms $f_\tau: M_0\to M_\tau$. Applying the above results to $M_\tau=\underline{\underline{M}}$, $M_{\tau+\Delta\tau}=\tilde M$ and $f_\tau=f$, $f_{\tau+\Delta\tau}=g\circ f$, assume that for $\Delta\tau=(0,\dots,0,\tau_i,0,\dots,0)$ there exist

$$\lim_{\tau_i\to 0}\frac{1}{\tau_i}(\tilde J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}-J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m})=\frac{\partial}{\partial\tau_i}J^{A_p\bar B_q}_{\Gamma_\ell\bar E_m},$$

$$\lim_{\tau_i\to 0}\frac{1}{\tau_i}(\tilde L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m}-L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m})=\frac{\partial}{\partial\tau_i}L^{A_p\bar B_q}_{\Gamma_\ell\bar E_m},$$

$$\lim_{\tau_i\to 0}\frac{1}{\tau_i}(\tilde\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v}-\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v})=\frac{\partial}{\partial\tau_i}\mu^{\Gamma_\ell\bar E_m}_{H_u\bar\Xi_v}$$

and

$$\lim_{\tau_i\to 0}\frac{1}{\tau_i}\underline{\underline{\mu}}^{H_u\bar\Xi_v}_{K_s\bar\Lambda_t}=\frac{\partial}{\partial\tau_i}\underline{\underline{\mu}}^{H_u\bar\Xi_v}_{K_s\bar\Lambda_t}.$$

Then, from eq. (3.1)

$$\frac{\partial}{\partial\tau_i}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}-L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})$$

$$=\sum_{\ell+m=p+q}\left\{\frac{\partial}{\partial\tau_i}(J^{A_p\bar{B}_q}_{\Gamma_\ell\bar{E}_m}+L^{A_p\bar{B}_q}_{\Gamma_\ell\bar{E}_m})\mu^{\Gamma_\ell\bar{E}_m}_{H_u\bar{\Xi}_v}+(J^{A_p\bar{B}_q}_{\Gamma_\ell\bar{E}_m}+L^{A_p\bar{B}_q}_{\Gamma_\ell\bar{E}_m})\frac{\partial}{\partial\tau_i}\mu^{\Gamma_\ell\bar{E}_m}_{H_u\bar{\Xi}_v}\right\}.$$

Thus, from eqs. (2.13) and (3.4)

$$\frac{\partial}{\partial\tau_i}\underline{\underline{\mu}}^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}=\frac{1}{2J}\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\left\{\frac{\partial}{\partial\tau_i}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}-L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})\overline{L^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v}}-\overline{(J^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v}-L^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v})}\frac{\partial}{\partial\tau_i}L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}\right\}$$

$$=\frac{1}{2J}\sum_{\substack{H_u,\Xi_v\\ u+v=p+q}}\left\{\frac{\partial}{\partial\tau_i}J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}\overline{L^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v}}-\overline{J^{K_s\bar{\Lambda}_t}_{H_u\bar{\Xi}_v}}\frac{\partial}{\partial\tau_i}L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}\right\}. \tag{3.5}$$

4. The Hilbert space of families of forms

Pull back due to diffeomorphism does not always preserve (p,q) forms. We need to treat them simultaneously for all p,q. Let $\tilde{F}^{p,q}$ denote the vector space of (p,q) forms and the direct sum by

$$\tilde{F}=\bigoplus_{\substack{0\le p\le n\\ 0\le q\le n}}\tilde{F}^{p,q}=\{\bigoplus\varphi^{(p,q)};\varphi^{(p,q)}\text{ is a }(p,q)\text{ form }\}.$$

We consider the vector space $\underline{\underline{\tilde{F}}}=\{\underline{\underline{\varphi}}=(\varphi_1,\varphi_2);\varphi_i\in\tilde{F}\}$ which satisfies

(1) $c\underline{\underline{\varphi}}=(c\varphi_1,c\varphi_2),\ c\varphi_i=\bigoplus c\varphi_i^{(p,q)}$

(2) $\underline{\underline{\varphi}}+\underline{\underline{\psi}}=(\varphi_1+\psi_1,\varphi_2+\psi_2),\ \varphi_i+\psi_i=\bigoplus(\varphi_i^{(p,q)}+\psi_i^{(p,q)})$,

(3) $\bar{\underline{\underline{\varphi}}}=(\bar{\varphi}_1,\bar{\varphi}_2),\ \bar{\varphi}_i=\bigoplus\overline{\varphi_i^{(p,q)}}$, where $^-$ denotes the complex conjugate,

(4) $*\underline{\underline{\varphi}}=(*\varphi_2,\sharp\varphi_1),\ \sharp\varphi_1=\bigoplus(-1)^{p+q+1}*\varphi_1^{(p,q)},\ \ *\varphi_2=\bigoplus*\varphi_2^{(p,q)}$.

We call the element $\underline{\underline{\varphi}}\in\underline{\underline{\tilde{F}}}$ a family of forms. Since $**\varphi_i^{(p,q)}=(-1)^{(p+q)}\varphi_i^{(p,q)}$, $*\sharp\varphi_i=\sharp*\varphi_i=\bigoplus(-1)^{p+q+1}**\varphi_i^{(p,q)}=-\varphi_i$. Therefore, $**\underline{\underline{\varphi}}=(*\sharp\varphi_1,\sharp*\varphi_2)=-\underline{\underline{\varphi}}$. It follows that

$$*(\underline{\underline{\varphi}}\pm i*\underline{\underline{\varphi}})=\mp i(\underline{\underline{\varphi}}\pm i*\underline{\underline{\varphi}}),$$

and $*\underline{\underline{\varphi}}=-i\underline{\underline{\varphi}}$ (*resp.* $*\underline{\underline{\varphi}}=-i\underline{\underline{\varphi}}$) if and only if $\underline{\underline{\varphi}}=(\varphi,i\sharp\varphi)$ (*resp.* $\underline{\underline{\varphi}}=(\varphi,-i\sharp\varphi)$).

We use the following operators on $\underline{\underline{\tilde{F}}}$ to $\underline{\underline{\tilde{F}}}$,

(1) $\partial\underline{\underline{\varphi}}=(\partial\varphi_1,\partial\varphi_2)$, where $\partial\varphi_i=\bigoplus\partial\varphi_i^{(p,q)}$,

$$\partial\varphi_{A_p,B_q}dz^{A_p}\wedge d\bar{z}^{B_q}=\frac{\partial}{\partial z^\alpha}\varphi_{A_p,B_q}dz^\alpha\wedge dz^{A_p}\wedge d\bar{z}^{B_q},$$

(2) $\bar{\partial}\underline{\underline{\varphi}} = (\bar{\partial}\varphi_1, \bar{\partial}\varphi_2)$, where $\bar{\partial}\varphi_i = \bigoplus \bar{\partial}\varphi_i^{(p,q)}$,

$$\bar{\partial}\varphi_{A_p,\bar{B}_q} dz^{A_p} \wedge d\bar{z}^{B_q} = \frac{\partial}{\partial \bar{z}^{\beta}} \varphi_{A_p,\bar{B}_q} d\bar{z}^{\beta} \wedge dz^{A_p} \wedge d\bar{z}^{B_q},$$

(3) $d\underline{\underline{\varphi}} = \partial\underline{\underline{\varphi}} + \bar{\partial}\underline{\underline{\varphi}}$.

We know $\partial\partial\varphi^{(p,q)} = \bar{\partial}\bar{\partial}\varphi(p,q) = 0$, $\bar{\partial}\partial\varphi^{(p,q)} = -\partial\bar{\partial}\varphi^{(p,q)}$. Hence $\partial\partial\underline{\underline{\varphi}} = \bar{\partial}\bar{\partial}\underline{\underline{\varphi}} = 0$, $\bar{\partial}\partial\underline{\underline{\varphi}} = -\partial\bar{\partial}\underline{\underline{\varphi}}$ and $dd\underline{\underline{\varphi}} = 0$. We say that $\underline{\underline{\varphi}}$ is closed if $\partial\underline{\underline{\varphi}} = 0$, $\underline{\underline{\varphi}}$ is coclosed if $d * \underline{\underline{\varphi}} = 0$ and $\underline{\underline{\varphi}}$ is harmonic if $\underline{\underline{\varphi}}$ is closed and coclosed. If $\underline{\underline{\varphi}}$ is harmonic, $\bar{\underline{\underline{\varphi}}}$ is also harmonic. A $\underline{\underline{\varphi}}$ is said to be analytic (resp. antianalytic) if it is closed and satisfies the condition $*\underline{\underline{\varphi}} = -i\underline{\underline{\varphi}}$ (resp. $*\underline{\underline{\varphi}} = i\underline{\underline{\varphi}}$). If $\underline{\underline{\varphi}}$ is analytic, $\underline{\underline{\varphi}}$ is coclosed and harmonic. Every harmonic family of forms can be uniquely written as the sum of an analytic and an antianalytic family of forms :

$$\underline{\underline{\varphi}} = \frac{1}{2}(\underline{\underline{\varphi}} + i * \underline{\underline{\varphi}}) + \frac{1}{2}(\underline{\underline{\varphi}} - i * \underline{\underline{\varphi}}).$$

These properties of families of forms are analogous to those of 1–forms on Riemann surfaces.

Now let

$$\underline{\underline{F}}^{\infty} = \{\underline{\underline{\varphi}} = (\varphi_1, \varphi_2) : \varphi_i^{(p,q)} \text{ are } C^{\infty} \text{ forms and } \sum_{i=1}^{2}\sum_{p,q}(\varphi_i^{(p,q)}, \varphi_i^{(p,q)}) < \infty\}.$$

For $\underline{\underline{\varphi}}, \underline{\underline{\psi}} \in \underline{\underline{F}}^{\infty}$

$$[\underline{\underline{\varphi}}, \underline{\underline{\psi}}] = \sum_{i=1}^{2}\sum_{p,q}(\varphi_i^{(p,q)}, \psi_i^{(p,q)})$$

has a finite value and is called the scalar product of φ and ψ. This satisfies

(1) $[c_1\underline{\underline{\varphi_1}} + c_2\underline{\underline{\varphi_2}}, \underline{\underline{\psi}}] = c_1[\underline{\underline{\varphi_1}}, \underline{\underline{\psi}}] + c_2[\underline{\underline{\varphi_2}}, \underline{\underline{\psi}}]$,

(2) $[\underline{\underline{\psi}}, \underline{\underline{\varphi}}] = \overline{[\underline{\underline{\varphi}}, \underline{\underline{\psi}}]}$,

(3) $[* \underline{\underline{\varphi}}, * \underline{\underline{\psi}}] = \sum_{p,q}(* \varphi_2^{(p,q)}, * \psi_2^{(p,q)}) + \sum_{p,q}(\sharp\varphi_1^{(p,q)}, \sharp\psi_1^{(p,q)}) = [\underline{\underline{\varphi}}, \underline{\underline{\psi}}]$,

where $\sharp\varphi_1^{(p,q)} = (-1)^{p+q+1} * \varphi_1^{(p,q)}$.

We note

$$[\underline{\underline{\varphi}} + i * \underline{\underline{\varphi}}, \underline{\underline{\psi}} - i * \underline{\underline{\psi}}] \;=\; [\underline{\underline{\varphi}}, \underline{\underline{\psi}}] - [* \underline{\underline{\varphi}}, * \underline{\underline{\psi}}] + i[* \underline{\underline{\varphi}}, \underline{\underline{\psi}}] + i[\underline{\underline{\varphi}}, * \underline{\underline{\psi}}] = 0$$

We then introduce some other subclasses of $\underline{\underline{F}}^{\infty}$,

$\underline{\underline{F}}_c^{\infty} = \{\underline{\underline{\varphi}} \in \underline{\underline{F}}^{\infty}; \underline{\underline{\varphi}} \text{ is closed }\}$, $*\underline{\underline{F}}_c^{\infty} = \{* \underline{\underline{\varphi}}; \underline{\underline{\varphi}} \in \underline{\underline{F}}_c^{\infty}\}$

$\underline{\underline{F}}_e^{\infty} = \{d\underline{\underline{\varphi}}; \underline{\underline{\varphi}} \in \underline{\underline{F}}^{\infty}\}$, $*\underline{\underline{F}}_e^{\infty} = \{* \underline{\underline{\varphi}}; \underline{\underline{\varphi}} \in \underline{\underline{F}}_e^{\infty}\}$,

$\underline{\underline{F}}_{eo}^{\infty} = \{d\underline{\underline{\varphi}}$; every $\varphi_i^{(p,q)}$ has a compact support },

$*\underline{\underline{F}}_{eo}^{\infty} = \{\underline{\underline{\varphi}} ; *\varphi \in \underline{\underline{F}}_{eo}^{\infty}\}$.

We now obtain some orthogonal relationships among these subclasses.

Lemma 4.1. *The class* $\underline{\underline{F}}_{C}^{\infty}$ *is orthogonal to* $*\underline{\underline{F}}_{eo}^{\infty}$, *i.e.* $[\underline{\underline{\varphi}}, *\underline{\underline{\psi}}] = 0$ *for* $\underline{\underline{\varphi}} \in \underline{\underline{F}}_{C}^{\infty}$ *and* $\underline{\underline{\psi}} \in \underline{\underline{F}}_{eo}^{\infty}$.

Proof. Using the partition of unity, we can assume that the support of $\underline{\underline{\psi}}$ is a Euclidean neighborhood of a point. Since $d\underline{\underline{\varphi}} = 0$, $\partial\varphi_i^{(p-1,q)} + \bar{\partial}\varphi_i^{(p,q-1)} = 0$. Therefore

$$d(\sum_{k=0}^{m}\varphi_i^{(k,m-k)}) = \partial\varphi_i^{(m,0)} + \bar{\partial}\varphi_i^{(0,m)} + \sum_{k=1}^{m-1}(\partial\varphi_i^{(k-1,m-k)} + \bar{\partial}\varphi_i^{(k,m-k-1)}) = 0.$$

By Poincare's lemma, [3) p.95] there exist $\tilde{\varphi}_i^{(k,m-k-1)}$ such that $\sum_{k=0}^{m}\varphi_i^{(k,m-k)}$ is represented locally by $d(\sum_{k=0}^{m-1}\tilde{\varphi}_i^{(k,m-k-1)})$. Hence, there is a $\underline{\underline{\tilde{\varphi}}}$ which satisfies $\underline{\underline{\varphi}} = d\underline{\underline{\tilde{\varphi}}}$ on the support of $\underline{\underline{\psi}}$. Write $\sum_{k=0}^{m-1}\tilde{\varphi}_i^{(k,m-k-1)} = \tilde{\varphi}_i^{(m-1)}$ and $\sum_{p+q=n-m}\psi_i^{(p,q)} = \psi_i^{(n-m)}$. We have

$$\begin{aligned}[\underline{\underline{\varphi}}, *\underline{\underline{\psi}}] &= \sum_{m}\sum_{p+q=m}\left\{(\varphi_1^{(p,q)}, *\psi_2^{(n-q,n-p)}) + (\varphi_2^{(p,q)}, \#\psi_1^{(n-q,n-p)})\right\} \\ &= \sum_{m}\left\{\iint d\tilde{\varphi}_1^{m-1}\wedge(-1)^m\overline{\psi_2^{n-m}} - \iint d\tilde{\varphi}_2^{m-1}\wedge(-1)^m\overline{\psi_1^{n-m}}\right\}.\end{aligned}$$

Since $d\overline{\psi_i^{n-m}} = 0$,

$$\begin{aligned}[\underline{\underline{\varphi}}, *\underline{\underline{\psi}}] &= \sum_m(-1)^m\left\{\iint_S d(\tilde{\varphi}_1^{m-1}\wedge\overline{\psi_2^{n-m}}) - \iint_S d(\tilde{\varphi}_2^{m-1}\wedge\overline{\psi_1^{n-m}})\right\} \\ &= \sum_m(-1)^m\left\{\iint_{\partial S}\tilde{\varphi}_1^{m-1}\wedge\overline{\psi_2^{n-m}} - \iint_{\partial S}\tilde{\varphi}_2^{m-1}\wedge\overline{\psi_1^{n-m}}\right\} = 0,\end{aligned}$$

where S is a regular region which contains the support of $\underline{\underline{\psi}}$.

The orthogonality is denoted by $\underline{\underline{F}}_{c}^{\infty} \perp *\underline{\underline{F}}_{eo}^{\infty}$. With this notation, it is clear that $*\underline{\underline{F}}_{c}^{\infty} \perp \underline{\underline{F}}_{eo}^{\infty}$.

Lemma 4.2. *If* $\underline{\underline{\varphi}} \in \underline{\underline{F}}^{\infty}$ *is orthogonal to every* $d\underline{\underline{\psi}} \in \underline{\underline{F}}_{eo}^{\infty}$, *then* φ *is coclosed.*

Proof. We have

$$\begin{aligned}[d\underline{\underline{\psi}}, \underline{\underline{\varphi}}] &= \sum_{i=1}^{2}\sum_{p,q}\iint(\partial\psi_i^{(p-1,q)} + \bar{\partial}\psi_i^{(p,q-1)})\wedge*\overline{\varphi_i^{(p,q)}} \\ &= \sum_{i=1}^{2}\sum_{m=1}^{2n}\iint d(\sum_{k=0}^{m-1}\psi_i^{(k,m-k-1)})\wedge\sum_{p+q=m}*\overline{\varphi_i^{(p,q)}} \\ &= \sum_{i=1}^{2}\sum_{m=1}^{2n}\iint\left\{d(\sum_{k=0}^{m-1}\psi_i^{(k,m-k-1)}\wedge\sum_{p+q=m}*\overline{\varphi_i^{(p,q)}})\right. \\ &\qquad \left. -(-1)^{(m-1)}\sum_{k=0}\psi_i^{(k,m-k-1)}\wedge d(\sum_{p+q=m}\overline{*\varphi_i^{(p,q)}})\right\}.\end{aligned}$$

Since the integral of the first term in { } vanishes, we get

$$\begin{aligned}[\underline{\underline{d\psi}},\underline{\underline{\varphi}}] &= \sum_{i=1}^{2}\sum_{m=1}^{2n}(-1)^m \iint \sum_{k=0}^{m-1} \psi_i^{(k,m-k-1)} \wedge d\Big(\sum_{p+q=m} * \overline{\varphi_i^{(p,q)}}\Big) \\ &= \sum_{i=1}^{2}\sum_{m=1}^{2n}(-1)^m \iint \sum_{k=0}^{m-1} \psi_i^{(k,m-k-1)} \wedge \big(\partial * \overline{\varphi_i^{(k+1,m-k-1)}} + \bar\partial * \overline{\varphi_i^{(k,m-k)}}\big).\end{aligned}$$

On the other hand, from the condition, $[d\underline{\underline{\psi}},\underline{\underline{\varphi}}]=0$ for every $\underline{\underline{\psi}}$ with compact support. Therefore, $\partial * \overline{\varphi_i^{(k+1,m-k-1)}} + \bar\partial * \overline{\varphi_i^{(k,m-k)}} = 0$. It follows that $d * \underline{\underline{\varphi}} = 0$ and $\underline{\underline{\varphi}}$ is coclosed.

From Lemma 4.2 and $[d\underline{\underline{\psi}}, * \underline{\underline{\varphi}}] = -[* d\underline{\underline{\psi}},\underline{\underline{\varphi}}]$, we have the following.

Lemma 4.3. *If $\underline{\underline{\varphi}} \in \underline{\underline{F}}^\infty$ is orthogonal to every $*d\underline{\underline{\psi}} \in * \underline{\underline{F}}_{eo}^\infty$, then $\underline{\underline{\varphi}}$ is closed.*

From these lemmas we obtain the following direct sum :

$$\underline{\underline{F}}_c^\infty \cap * \underline{\underline{F}}_c^\infty \dot{+} \underline{\underline{F}}_{eo}^\infty \dot{+} * \underline{\underline{F}}_{eo}^\infty.$$

Let $\underline{\underline{F}}, \underline{\underline{F}}_e, * \underline{\underline{F}}_e, \underline{\underline{F}}_{eo}, * \underline{\underline{F}}_{eo}$ be the completion of $\underline{\underline{F}}^\infty, \underline{\underline{F}}_e^\infty, * \underline{\underline{F}}_e^\infty, \underline{\underline{F}}_{eo}^\infty, * \underline{\underline{F}}_{eo}^\infty$, respectively. The element in $\underline{\underline{F}}$ is called a family of square summable forms (currents). Using Weyl and Kodaira's lemma ; "If $\underline{\underline{\varphi}} \in \underline{\underline{F}}$ is orthogonal to $\underline{\underline{F}}_{eo}$ and $*\underline{\underline{F}}_{eo}$, then $\underline{\underline{\varphi}} \in \underline{\underline{F}}^\infty$", [2), 8)] the orthogonal decomposition of $\underline{\underline{F}}$ follows

$$\underline{\underline{F}} = \underline{\underline{F}}_c^\infty \cap * \underline{\underline{F}}_c^\infty \dot{+} \underline{\underline{F}}_{eo} \dot{+} * \underline{\underline{F}}_{eo}. \tag{4.1}$$

Write the subspaces

$$\underline{\underline{F}}_h = \underline{\underline{F}}_c^\infty \cap * \underline{\underline{F}}_c^\infty, \underline{\underline{F}}_c = \underline{\underline{F}}_h \dot{+} \underline{\underline{F}}_{eo}, * \underline{\underline{F}}_c = \underline{\underline{F}}_h \dot{+} * \underline{\underline{F}}_{eo}.$$

We can verify $\underline{\underline{F}}_c$ is the completion of $\underline{\underline{F}}_c^\infty$. We denote the orthogonal complement of $*\underline{\underline{F}}_e$ in $\underline{\underline{F}}$ by $\underline{\underline{F}}_{co}$ and set $\underline{\underline{F}}_{he} = \underline{\underline{F}}_h \cap \underline{\underline{F}}_e$, $\underline{\underline{F}}_{ho} = \underline{\underline{F}}_h \cap \underline{\underline{F}}_{co}$. Then we have

$$\underline{\underline{F}}_e = \underline{\underline{F}}_{he} \dot{+} \underline{\underline{F}}_{eo}, \underline{\underline{F}}_{co} = \underline{\underline{F}}_{ho} \dot{+} \underline{\underline{F}}_{eo}$$

and

$$\underline{\underline{F}} = \underline{\underline{F}}_e \dot{+} * \underline{\underline{F}}_{ho} = \underline{\underline{F}}_{he} \dot{+} * \underline{\underline{F}}_{ho} \dot{+} \underline{\underline{F}}_{eo} \dot{+} * \underline{\underline{F}}_{eo}. \tag{4.2}$$

We denote the subspace of analytic families of forms by

$$\underline{\underline{F}}_a = \{\underline{\underline{\varphi}} \in \underline{\underline{F}}_h ; * \underline{\underline{\varphi}} = - i\underline{\underline{\varphi}}\} \text{ and } \bar{\underline{\underline{F}}}_a = \{\bar{\underline{\underline{\varphi}}} ; \underline{\underline{\varphi}} \in \underline{\underline{F}}_a\}.$$

5. Distortions of families of forms by pull back

For a diffeomorphism f from M to $\tilde M$ and $\underline{\underline{\varphi}} \in F^\infty(\tilde M)$, the pull back is defined by $\underline{\underline{\varphi}} \circ f = (\varphi_1 \circ f, \varphi_2 \circ f)$, where the (s,t) component of $\varphi_i \circ f$ is

$$\sum_{p+q=s+t} \varphi_{iA_p\bar{B}_q} J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} dz^{K_s} \wedge d\bar{z}^{\Lambda_t}.$$

The norm of $\underline{\underline{\varphi}}$ is denoted by $\| \underline{\underline{\varphi}} \| = \sqrt{[\underline{\underline{\varphi}}, \underline{\underline{\varphi}}]}$. As for the norm of pull back, in a similar manner to Theorem 3.1 we have the following

Theorem 5.1. *Let μ_m for f satisfy that $\|\mu_m\| \le k_m^2 \le k^2 < 1$. Then*

$$\|\underline{\underline{\varphi}} \circ f\|^2 + \|(*\underline{\underline{\varphi}}) \circ f\|^2 \le 2\,{}_{2n}C_n \frac{1+k^2}{1-k^2} \|\underline{\underline{\varphi}}\|^2 \text{ for } \varphi \in \underline{\underline{F}} \tag{5.1}$$

$$\|\underline{\underline{\psi}} \circ f - i * (\underline{\underline{\psi}} \circ f)\| \le k \|\underline{\underline{\psi}} \circ f + i * (\underline{\underline{\psi}} \circ f)\| \text{ for } \psi \in \underline{\underline{F}}_a. \tag{5.2}$$

Proof. Using usual normalization of variables,

$$\|\underline{\underline{\varphi}} \circ f\|^2 + \|(*\underline{\underline{\varphi}}) \circ f\|^2$$

$$= \sum_{i=1}^{2} \sum_{m=0}^{2n} \sum_{\substack{K_s, \Lambda_t \\ s+t=m}} \iint \Big\{ \Big| \sum_{p+q=m} \varphi_{iA_p\bar{B}_q} J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} \Big|^2 + \Big| \sum_{p+q=m} \varphi_{iA_p\bar{B}_q} L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} \Big|^2 \Big\} \frac{\omega^n}{n!}$$

$$\le \sum_{i=1}^{2} \sum_{m=0}^{2n} \iint \Big\{ \sum_{\substack{A_p B_q \\ p+q=m}} |\varphi_{iA_p B_q}|^2 \sum_{\substack{K_s, \Lambda_t \\ s+t=m}} \sum_{\substack{A_p, B_q \\ p+q=m}} \big(|J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2 + |L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}|^2 \big) \Big\} \frac{\omega^n}{n!}$$

$$\le \sum_{i=1}^{2} \sum_{M=0}^{2n} \iint \Big\{ \sum_{\substack{A_p, B_q \\ p+q=m}} |\varphi_{iA_p B_q}|^2 \sum_{\substack{A_p, B_q \\ p+q=m}} 2 \frac{1+\|\mu_m\|}{1-\|\mu_m\|} \Big\} \frac{\bar{\omega}^n}{n!}$$

$$\le 2\,{}_{2n}C_n \frac{1+k^2}{1-k^2} \|\underline{\underline{\varphi}}\|^2.$$

The $\underline{\underline{\psi}} \in \underline{\underline{F}}_a$ is written as $\underline{\underline{\psi}} = (\psi_1, i\sharp\psi_1)$ and

$$\underline{\underline{\psi}} \circ f + i * (\underline{\underline{\psi}} \circ f) = (\psi_1 \circ f - * ((\sharp\psi_1) \circ f), i((\sharp\psi_1) \circ f + \sharp(\psi_1 \circ f))),$$

$$\underline{\underline{\psi}} \circ f - i * (\underline{\underline{\psi}} \circ f) = (\psi_1 \circ f + * ((\sharp\psi_1) \circ f), i((\sharp\psi_1) \circ f - \sharp(\psi_1 \circ f))).$$

Denote

$$\psi_1 \circ f - * ((\sharp\psi_1) \circ f) = \bigoplus \Psi_{K_s\bar{\Lambda}_t} dz^{K_s} \wedge d\bar{z}^{\Lambda_t},$$

$$\psi_1 \circ f + * ((\sharp\psi_1) \circ f) = \bigoplus \Phi_{K_s\bar{\Lambda}_t} dz^{K_s} \wedge d\bar{z}^{\Lambda_t},$$

where

$$\Psi_{K_s\bar{\Lambda}_t} = \sum_{p+q=s+t} \psi_{1A_p\bar{B}_q} (J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} + L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}),$$

$$\Phi_{K_s\bar{\Lambda}_t} = \sum_{p+q=s+t} \psi_{1A_p\bar{B}_q} (J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t} - L^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}) = \sum_{p+q=s+t} \psi_{1A_p\bar{B}_q} \Big\{ \sum_{u+v=s+t} (J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v} + L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}) \mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t} \Big\}.$$

Remarking that $\Phi_{K_s\bar{\Lambda}_t}=\sum_{u+v=s+t}\Psi_{H_u\bar{\Xi}_v}\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}$, it follows

$$\begin{aligned}\|\underline{\underline{\psi}}\circ f-i*(\underline{\underline{\psi}}\circ f)\|^2 &= 2\sum_{m=0}^{2n}\sum_{\substack{K_s,\Lambda_t\\ s+t=m}}\iint|\Phi_{K_s\bar{\Lambda}_t}|^2\frac{\omega^n}{n!}\\ &\le 2\sum_{m=0}^{2n}\sum_{\substack{K_s,\Lambda_t\\ s+t=m}}\iint\Big\{\sum_{\substack{H_u,\Xi_v\\ u+v=m}}|\Psi_{H_u\bar{\Xi}_v}|^2\sum_{\substack{H_u,\Xi_v\\ u+v=m}}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}|^2\Big\}\frac{\omega^n}{n!}\\ &\le k^2\|\underline{\underline{\psi}}\circ f+i*(\underline{\underline{\psi}}\circ f)\|^2.\end{aligned}$$

Similar to Lemma 2.1 we have

Lemma 5.1. $[\underline{\underline{\varphi}}\circ f,-*((*\underline{\underline{\psi}})\circ f)]=[\underline{\underline{\varphi}},\underline{\underline{\psi}}]$

Proof. Let $\underline{\underline{\psi}}=(\psi_1,\psi_2)$. Then $-*((*\underline{\underline{\psi}})\circ f)=(-*((\#\psi_1)\circ f),-\#((*\psi_2)\circ f)))$.

Hence

$$\begin{aligned}&[\underline{\underline{\varphi}}\circ f,-*((*\underline{\underline{\psi}})\circ f)]\\ &=\sum_{i=1}^{2}\sum_{m=0}^{2n}\sum_{p+q=m}\sum_{u+v=m}\sum_{s+t=m}(\varphi_{iA_p\bar{B}_q}J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}dz^{K_s}\wedge d\bar{z}^{\Lambda_t},\psi_{iH_u\bar{\Xi}_v}L^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}dz^{K_s}\wedge d\bar{z}^{\Lambda_t})\\ &=\sum_{i=1}^{2}\sum_{m=0}^{2n}\sum_{\substack{A_p,B_q\\ p+q=m}}\sum_{\substack{H_u,\Xi_v\\ u+v=m}}\iint\Big\{\varphi_{iA_p\bar{B}_q}\overline{\psi_{iH_u\bar{\Xi}_v}}\sum_{\substack{K_s,\Lambda_t\\ s+t=m}}J^{A_p\bar{B}_q}_{K_s\bar{\Lambda}_t}\overline{L^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}}\Big\}\frac{\omega^n}{n!}\\ &=\sum_{i=1}^{2}\sum_{m=0}^{2n}\sum_{\substack{A_p,B_q\\ p+q=m}}\iint\varphi_{iA_p\bar{B}_q}\overline{\psi_{iA_p\bar{B}_q}}\frac{\tilde{\omega}^n}{n!}=[\underline{\underline{\varphi}},\underline{\underline{\psi}}].\end{aligned}$$

Let $f^{\#}$ be a linear mapping from $\underline{\underline{F}}(M)$ to $\underline{\underline{F}}(\tilde{M})$ which is defined by $f^{\#}(\underline{\underline{\varphi}})=\underline{\underline{\varphi}}\circ f^{-1}$. It is clear that $(f^{-1})^{\#}\circ f^{\#}$ and $f^{\#}\circ(f^{-1})^{\#}$ are identity mappings.

Lemma 5.2. *The $f^{\#}$ preserves the subspaces $\underline{\underline{F}}_{eo}$, $\underline{\underline{F}}_{e}$ and $\underline{\underline{F}}_{c}$ i.e.*

1) $f^{\#}(\underline{\underline{F}}_{eo}(M))=\underline{\underline{F}}_{eo}(\tilde{M})$, 2) $f^{\#}(\underline{\underline{F}}_{e}(M))=\underline{\underline{F}}_{e}(\tilde{M})$, 3) $f^{\#}(\underline{\underline{F}}_{e}(M))=\underline{\underline{F}}_{c}(\tilde{M})$.

Proof. As to 1) and 2), note that $(d\underline{\underline{\varphi}})\circ f=d(\underline{\underline{\varphi}}\circ f)$. For $\underline{\underline{\varphi}}\in\underline{\underline{F}}_{c}(M)$ and $\underline{\underline{\psi}}\in\underline{\underline{F}}_{eo}(\tilde{M})$, from Lemma 5.1, $[\underline{\underline{\varphi}}\circ f^{-1},*\underline{\underline{\psi}}]=[\underline{\underline{\varphi}},*(\underline{\underline{\psi}}\circ f)]=0$. This follows 3).

Let $\underline{\underline{F}}_{1x}$ be a subspace of $\underline{\underline{F}}_{h}$ whose element has a vanishing second part as $(\varphi_1,0)$ and $\underline{\underline{F}}_{1x^{\perp}}=\{\underline{\underline{\psi}}=(\psi_1,0);[\underline{\underline{\psi}},\underline{\underline{\varphi}}]=0$ for every $\underline{\underline{\varphi}}\in\underline{\underline{F}}_{1x}\}$. Set $\underline{\underline{F}}_{x}=\underline{\underline{F}}_{1x}+*\underline{\underline{F}}_{1x^{\perp}}$. Since $*\underline{\underline{F}}_{x}=\underline{\underline{F}}_{1x^{\perp}}+*\underline{\underline{F}}_{1x}$, $*\underline{\underline{F}}_{x}$ is orthogonal to $\underline{\underline{F}}_{x}$ and $\underline{\underline{F}}_{x}\dotplus*\underline{\underline{F}}_{x}=\underline{\underline{F}}_{h}$. A subspace $\underline{\underline{F}}_{x}$ of $\underline{\underline{F}}_{h}$ is called a behavior space if $\underline{\underline{F}}_{x}\dotplus*\underline{\underline{F}}_{x}=\underline{\underline{F}}_{h}$.

For example, $\underline{\underline{F}}_{h}=\{\underline{\underline{\varphi}}=(\varphi_1,0)\in\underline{\underline{F}}_{h}\}$ and $\underline{\underline{F}}_{he}=\{(\varphi_1,\varphi_2);(\varphi_1,0)\in\underline{\underline{F}}_{he},(0,\varphi_2)\in\underline{\underline{F}}_{ho}\}$ are behavior spaces. Let P be the projection from $\underline{\underline{F}}$ to $\underline{\underline{F}}_{h}$. For a behavior space $\underline{\underline{F}}_{x}(M)$ set $\underline{\underline{F}}_{x}(\tilde{M})=\{P\circ f^{\#}(\underline{\underline{\varphi}});\underline{\underline{\varphi}}\in\underline{\underline{F}}_{x}(M)\}$.

Lemma 5.3. *The* $\underline{\underline{F}}_x(\tilde{M})$ *is a behavior space on* $\tilde{M}$, *i.e.*

$$\underline{\underline{F}}_x(\tilde{M}) \dotplus * \underline{\underline{F}}_x(\tilde{M}) = \underline{\underline{F}}_h(\tilde{M}).$$

Proof. For $\underline{\underline{\varphi}}, \underline{\underline{\psi}} \in \underline{\underline{F}}_x(M)$, $f^{\#}(\underline{\underline{\varphi}})$ and $f^{\#}(\underline{\underline{\psi}})$ are closed. Hence, from (4.1),

$$[P \circ f^{\#}(\underline{\underline{\varphi}}), *(P \circ f^{\#}(\underline{\underline{\psi}}))] = [f^{\#}(\underline{\underline{\varphi}}), *f^{\#}(\underline{\underline{\psi}})].$$

Using Lemma 5.1,

$$[f^{\#}(\underline{\underline{\varphi}}), *f^{\#}(\underline{\underline{\psi}})] = [\underline{\underline{\varphi}}, *\underline{\underline{\psi}}] = 0.$$

Therefore, $\underline{\underline{F}}_x(\tilde{M})$ is orthogonal to $*\underline{\underline{F}}_x(\tilde{M})$. On the other hand, suppose a $\underline{\underline{\tilde{\varphi}}} \in \underline{\underline{F}}_h(\tilde{M})$ is orthogonal to $\underline{\underline{F}}_x(\tilde{M}) + * \underline{\underline{F}}_x(\tilde{M})$. For every $\underline{\underline{\psi}} \in \underline{\underline{F}}_x(M)$, from Lemma 5.1,

$$0 = [\underline{\underline{\tilde{\varphi}}}, *f^{\#}(\underline{\underline{\psi}})] = [\underline{\underline{\tilde{\varphi}}} \circ f, *\underline{\underline{\psi}}] = [P \circ (f^{-1})^{\#}(\underline{\underline{\tilde{\varphi}}}), *\underline{\underline{\psi}}].$$

Hence, $P \circ (f^{-1})^{\#}(\tilde{\varphi}) \in \underline{\underline{F}}_x(M)$ and $(f^{-1})^{\#}(\tilde{\varphi}) \in \underline{\underline{F}}_x(M) + \underline{\underline{F}}_{eo}(M)$. It follows that $\tilde{\varphi} \in \underline{\underline{F}}_x(\tilde{M})$. Thus $\underline{\underline{\tilde{\varphi}}}$ vanishes.

Lemma 5.4. *Let* $\underline{\underline{\psi}} \in \underline{\underline{F}}_a(M)$ *and* $\underline{\underline{\tilde{\psi}}} \in \underline{\underline{F}}_a(\tilde{M})$ *satisfy* $\underline{\underline{\tilde{\psi}}} \circ f - \underline{\underline{\psi}} \in \underline{\underline{F}}_x(M) + \underline{\underline{F}}_{eo}(M)$. *Then*

$$\frac{1}{2}\| \underline{\underline{\tilde{\psi}}} \circ f + i * (\underline{\underline{\tilde{\psi}}} \circ f) \| \le \frac{1}{1-k} \| \underline{\underline{\psi}} \|, \tag{5.3}$$

$$\frac{1}{2}\| \underline{\underline{\tilde{\psi}}} \circ f - i * (\underline{\underline{\tilde{\psi}}} \circ f) \| \le \frac{k}{1-k} \| \underline{\underline{\tilde{\psi}}} \|, \tag{5.4}$$

$$\frac{1}{2}\| \underline{\underline{\tilde{\psi}}} \circ f + i * (\underline{\underline{\tilde{\psi}}} \circ f) - 2\underline{\underline{\psi}} \| \le \frac{k}{1-k} \| \underline{\underline{\psi}} \|, \tag{5.5}$$

where μ_m *for* f *satisfies that* $\|\mu_m\| = k_m^2 \le k^2 < 1$.

Proof. Let $\Psi = (\underline{\underline{\tilde{\psi}}} \circ f + i * (\underline{\underline{\tilde{\psi}}} \circ f))/2, \Phi = (\underline{\underline{\tilde{\psi}}} \circ f - i * (\underline{\underline{\tilde{\psi}}} \circ f))/2$.

Under these conditions, observe that

$$0 = [\underline{\underline{\tilde{\psi}}} \circ f - \underline{\underline{\psi}}, *(\underline{\underline{\tilde{\psi}}} \circ f - \underline{\underline{\psi}})] = [\Psi - \underline{\underline{\psi}} + \Phi, *(\Psi - \underline{\underline{\psi}} + \Phi)].$$

Since $*(\Psi - \underline{\underline{\psi}}) = -i(\Psi - \underline{\underline{\psi}})$ and $*\Phi = i\Phi$, we have $\|\Psi - \underline{\underline{\psi}}\| = \|\Phi\|$. From Theorem 5.1 it follows that $\|\Phi\| \le k\|\Psi\|$. Therefore

$$\|\Psi\| \le \frac{1}{1-k} \|\underline{\underline{\psi}}\| \text{ and } \|\Phi\| \le \frac{k}{1-k} \|\underline{\underline{\psi}}\|.$$

6. Ahlfors-Rauch type variational formulas

Let M_τ be an n–dimensional complex manifold with a parameter $\tau=(\tau_1,\tau_2,\dots,\tau_m)$ in a neighborhood of zero in m dimensional Euclidean space R^m and f_τ be a diffeomorphism from M_0 to M_τ whose Beltrami tensor $\mu_{K_s\bar{\Lambda}_t}^{H_u\bar{\Xi}_v}(z,\tau)$ satisfies

(1) $\|\mu_m\|=\sup\sum_{\substack{H_u,\Xi_v\\u+v=m}}\sum_{\substack{K_s,\Lambda_t\\s+t=m}}|\mu_{K_s\bar{\Lambda}_t}^{H_u\bar{\Xi}_v}|^2\le k_m(\tau)^2\le k(\tau)^2<1$

(2) $\mu_{K_s\bar{\Lambda}_t}^{H_u\bar{\Xi}_v}(z,\tau)$ is continuously differentiable with respect to τ for almost all $z\in M_0$ and $\frac{\partial}{\partial\tau_\ell}\mu_{K_s\bar{\Lambda}_t}^{H_u\bar{\Xi}_v}(z,\tau)$ is bounded and measurable.

Theorem 6.1 *Let $\underline{\underline{F}}_x(M_0)$ be a behavior space and analytic families of forms $\underline{\underline{\varphi}}^\tau,\underline{\underline{\psi}}^\tau$ on M_τ satisfy $\underline{\underline{\varphi}}^\tau\circ f_\tau-\underline{\underline{\varphi}}^0$, $\underline{\underline{\psi}}^\tau\circ f_\tau-\underline{\underline{\psi}}^0\in\underline{\underline{F}}_x(M_0)+\underline{\underline{F}}_{eo}(M_0)$. Assume that $\lim_{\tau\to0}\frac{k(\tau)^2}{|\tau|}=0$. Then*

$$\frac{\partial}{\partial\tau_\ell}[\underline{\underline{\psi}}^\tau\circ f_\tau-\underline{\underline{\psi}}^0,*\overline{\underline{\underline{\varphi}}^0}]|_{\tau=0}$$
$$=\sum_{m=0}^{2n}\sum_{s+t=m}\sum_{u+v=m}4(-1)^{s+t+t(n-s)}\iint_{M_0}\psi_{1H_u\bar{\Xi}_v}^0\varphi_{2K_{n-s}\bar{\Lambda}_{n-t}}^0\frac{\partial}{\partial\tau_\ell}\mu_{K_s\bar{\Lambda}_t}^{H_u\bar{\Xi}_v}(z,0)$$
$$\times dz^{K_s}\wedge dz^{K_{n-s}}\wedge d\bar{z}^{\Lambda_t}\wedge d\bar{z}^{\Lambda_{n-t}},$$

where $\underline{\underline{\varphi}}^\tau=(\varphi_1^\tau,\varphi_2^\tau)$, $\varphi_i^\tau=\bigoplus\varphi_{iA_pB_q}^\tau dz^{A_p}\wedge d\bar{z}^{B_q}$, $\underline{\underline{\psi}}^\tau=(\psi_1^\tau,\psi_2^\tau)$, $\psi_i^\tau=\bigoplus\psi_{iA_pB_q}^\tau dz^{A_p}\wedge d\bar{z}^{B_q}$.

Proof. Observe that

$$\begin{aligned}[\underline{\underline{\psi}}^\tau\circ f_\tau-\underline{\underline{\psi}}^0,*\overline{\underline{\underline{\varphi}}^0}] &= [\tfrac{1}{2}(\underline{\underline{\psi}}^\tau\circ f_\tau+i*(\underline{\underline{\psi}}^\tau\circ f_\tau))-\underline{\underline{\psi}}^0+\tfrac{1}{2}(\underline{\underline{\psi}}^\tau\circ f_\tau-i*(\underline{\underline{\psi}}^\tau\circ f_\tau)),*\overline{\underline{\underline{\varphi}}^0}]\\ &= \tfrac{1}{2}[\underline{\underline{\psi}}^\tau\circ f_\tau-i*(\underline{\underline{\psi}}^\tau\circ f_\tau),*\overline{\underline{\underline{\varphi}}^0}].\end{aligned}$$

Since $\underline{\underline{\varphi}}^\tau$ and $\underline{\underline{\psi}}^\tau$ belong to $\underline{\underline{F}}_a$, we have

$$\begin{aligned}\underline{\underline{\psi}}^\tau\circ f_\tau-i*(\psi^\tau\circ f_\tau) &= (\psi_1^\tau\circ f_\tau-i*(\psi_2^\tau\circ f_\tau),\psi_2^\tau\circ f_\tau-i\#(\psi_1^\tau\circ f_\tau))\\ &= (\psi_1^\tau\circ f_\tau+*((\#\psi_1^\tau)\circ f_\tau)),-i\#(\psi_1^\tau\circ f_\tau+*((\#\psi_1^\tau)\circ f_\tau))),\\ *\overline{\underline{\underline{\varphi}}^0}=(*\overline{\varphi_2^0},\#\overline{\varphi_1^0}) &= (*\overline{\varphi_2^0},-i\#(*\overline{\varphi_2^0})).\end{aligned}$$

Hence, $[\underline{\underline{\psi}}^\tau\circ f_\tau-\underline{\underline{\psi}}^0,*\overline{\underline{\underline{\varphi}}^0}]=2(\psi_1^\tau\circ f_\tau+*((\#\psi_1^\tau)\circ f_\tau),*\overline{\varphi_2^0})$. By (2.7) and (3.1)

$$\begin{aligned}\psi_1^\tau\circ f_\tau+*((\#\psi_1^\tau)\circ f_\tau) &= \sum_{m=0}^{2n}\sum_{p+q=m}\sum_{s+t=m}\psi_{1A_pB_q}^\tau(J_{K_s\bar{\Lambda}_t}^{A_p\bar{B}_q}-L_{K_s\bar{\Lambda}_t}^{A_p\bar{B}_q})dz^{K_s}\wedge d\bar{z}^{\Lambda_t}\\ &= \sum_{m=0}^{2n}\sum_{p+q=m}\sum_{s+t=m}\psi_{1A_pB_q}^\tau(J_{H_u\bar{\Xi}_v}^{A_p\bar{B}_q}+L_{H_u\bar{\Xi}_v}^{A_p\bar{B}_q})\mu_{K_s\bar{\Lambda}_t}^{H_u\bar{\Xi}_v}dz^{K_s}\wedge d\bar{z}^{\Lambda_t}.\end{aligned}$$

We can write this as

$$\sum_{m=0}^{2n}\sum_{p+q=m}\sum_{s+t=m}\{\psi^{\tau}_{1A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}+L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})-2\psi^{0}_{1H_u\bar{\Xi}_v}\}\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}dz^{K_s}\wedge d\bar{z}^{\Lambda_t}$$
$$+\sum_{m=0}^{2n}\sum_{p+q=m}\sum_{s+t=m}2\psi^{0}_{1H_u\bar{\Xi}_v}\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}dz^{K_s}\wedge d\bar{z}^{\Lambda_t}.$$

Denote the first term by $\psi'(\tau)$, then

$$\|\psi'(\tau)\|^2 = \sum_{m=0}^{2n}\sum_{\substack{K_s,\Lambda_t\\ s+t=m}}\iint_{M_0}|\sum_{p+q=m}\{\psi^{\tau}_{1A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}+L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})-2\psi^{0}_{1H_u\bar{\Xi}_v}\}\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}|^2\frac{\omega^n}{n!}$$
$$\leq \sum_{m=0}^{2n}\sum_{\substack{K_s,\Lambda_t\\ s+t=m}}\iint_{M_0}[\sum_{\substack{H_u,\Xi_v\\ u+v=m}}|\sum_{p+q=m}\{\psi^{\tau}_{1A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}+L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})-2\psi^{0}_{1H_u\bar{\Xi}_v}\}|^2$$
$$\times\sum_{\substack{H_u,\Xi_v\\ u+v=m}}|\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}|^2]\frac{\omega^n}{n!}$$
$$\leq \sum_{m=0}^{2n}k_m(\tau)^2\iint_{M_0}[\sum_{\substack{H_u,\Xi_v\\ u+v=m}}|\sum_{p+q=m}\{\psi^{\tau}_{1A_p\bar{B}_q}(J^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v}+L^{A_p\bar{B}_q}_{H_u\bar{\Xi}_v})-2\psi^{0}_{1H_u\bar{\Xi}_v}\}|^2]\frac{\omega^n}{n!}$$
$$\leq \frac{k(\tau)^2}{2}\|\underline{\underline{\psi}}^{\tau}\circ f_\tau+i*(\underline{\underline{\psi}}^{\tau}\circ f_\tau)-2\underline{\underline{\psi}}^{0}\|^2\leq\frac{2k(\tau)^4}{(1-k(\tau))^2}\|\underline{\underline{\psi}}^{0}\|^2$$

Thus

$$\lim_{\tau_\ell\to 0}\frac{1}{\tau_\ell}[\underline{\underline{\psi}}^{\tau}\circ f_\tau-\underline{\underline{\psi}}^{0},*\overline{\underline{\underline{\psi}}^{0}}]\qquad(\tau=(0,\dots,\tau_\ell,\dots,0))$$
$$=\lim_{\tau_\ell\to 0}\sum_{m=0}^{2n}\sum_{s+t=m}\sum_{u+v=m}4(-1)^{s+t+t(n-s)}\iint_{M_0}\{\psi^{0}_{1H_u\bar{\Xi}_v}\varphi^{0}_{2K_{n-s}\bar{\Lambda}_{n-t}}\times\frac{1}{\tau_\ell}\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}(z,\tau)\}$$
$$dz^{K_s}\wedge dz^{K_{n-s}}\wedge d\bar{z}^{\Lambda_t}\wedge d\bar{z}^{\Lambda_{n-t}}+\lim_{\tau_\ell\to 0}\frac{1}{\tau_\ell}(\psi'(\tau),*\overline{\varphi^{0}_{2}})$$
$$=\sum_{m=0}^{2n}\sum_{s+t=m}\sum_{u+v=m}4(-1)^{s+t+t(n-s)}\iint_{M_0}\{\psi^{0}_{1H_u\bar{\Xi}_v}\varphi^{0}_{2K_{n-s}\bar{\Lambda}_{n-t}}\frac{\partial}{\partial\tau_\ell}\mu^{H_u\bar{\Xi}_v}_{K_s\bar{\Lambda}_t}(z,0)\}$$
$$dz^{K_s}\wedge dz^{K_{n-s}}\wedge d\bar{z}^{\Lambda_t}\wedge d\bar{z}^{\Lambda_{n-t}}.$$

Corollary 6.1. *If ψ^{τ}_{1} and φ^{τ}_{2} in Theorem 6.1 are $(n,0)$-holomorphic forms, then* $\frac{\partial}{\partial\tau_\ell}[\underline{\underline{\psi}}^{\tau}\circ f_\tau-\underline{\underline{\psi}}^{0},*\overline{\underline{\underline{\varphi}}^{0}}]|_{\tau=0}=4\iint_{M_0}\psi^{0}_{1}\varphi^{0}_{2}\frac{\partial}{\partial\tau_\ell}\mu^{12\cdots n}_{\bar{1}\bar{2}\cdots\bar{n}}(z,0)\frac{\omega^n}{n!}$.

Department of Mechanical and System Engineering (F. M.),
Faculty of Engineering and Design,
Kyoto Institute of Technology,
Matsugasaki, Sakyo-ku, Kyoto 606-8585

References

1) L.V. Ahlfors and L. Sario, Riemann surfaces., Princeton (1960).
2) L.V. Ahlfors, Analytic functions. Proceedings (Princeton 1957), 45–66, (1960).
3) W.V.D. Hodge, The theory and applications to harmonic integrals., Cambridge University Press (1952).
4) K. Kodaira, *Annals of Math.*, **50**, 587–665 (1949).
5) K. Kodaira, Complex manifolds and deformation of complex structures., Springer-Verlag 1985.
6) Y. Kusunoki and F. Maitani, *Math. Z.*, **181**, 435–450, (1982).
7) F. Maitani, *J. Math. Kyoto Univ.*, **24**, 49–66, (1984).
8) H.E. Rauch, *Proc. Nat. Acd. Sci.*, **41**, 42–49, (1955).
9) H.E. Rauch, *Bull. Amer. Math. Soc.*, **71**, 1–39, (1965).
10) G. de Rham, Differentiable Manifolds., Springer-Verlag 1984.