Memoirs of the Faculty of Engineering and Design
Kyoto Institute of Technology 47 (February, 1999)

A Note on Möbius Transformations in Space

Xiantao WANG, Weiqi YANG, Fumio MAITANI and Yueping JIANG

(Received August 14, 1998; Accepted September 18, 1998)

Abstract

In this paper, type criteria for Möbius transformations in $\bar{\mathbf{R}}^n$, which only depend on coefficients in the Möbius transformation expression, are given. In addition, a chordal distortion theorem is established.

Key Words: *Möbius transformation, Clifford algebra, Type criteria, Distortion theorem.*

1. Introduction

Let $\mathbf{R}^n$ be n-dimensional Euclidean space, $\bar{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ be the one point compactification of $\mathbf{R}^n$ and $M(\bar{\mathbf{R}}^n)$ be the Möbius group of all orientation-preserving Möbius transformations in $\bar{\mathbf{R}}^n$. Using the Clifford matrix representation of Möbius transformations in high dimensions, Ahlfors[1),2)] discussed the classification and type criteria of elements in $M(\bar{\mathbf{R}}^n)$ for $n \geq 3$. Fang, Liu, Wang and other authors improved on Ahlfors' results.[4)~8)] In this paper, we continue this study. Type criteria for Möbius transformations in $\bar{\mathbf{R}}^n$, which only depend on coefficients in the Möbius transformation expression, are given. In addition, a chordal distortion theorem is established.

2. Preliminary material

We require the following preliminary material.

Let A_n denote the associative algebra over the real numbers generated by $1, e_1, e_2, \ldots, e_{n-1}$ subject to the relations

$$e_i^2 = -1,\ e_i e_j = -e_j e_i\,(i \neq j),\ i, j = 1, 2, \ldots, n-1. \tag{2.1}$$

For any $a \in A_n$, there is a unique representation in the form

$$a = a_0 + \sum a_v E_v, \tag{2.2}$$

where a_0 and a_v are real, the summation is over all multi-indices $v = (v_1, v_2, \ldots, v_p)$ with $0 < v_1 < v_2 \ldots < v_p \leq n-1$, and $E_v = e_{v_1} e_{v_2} \ldots e_{v_p}$. a_0 is said to be the real part of a denoted by $a_0 = Re(a)$. The modulus of a is defined by

$$|a| = \left(a_0^2 + \sum a_v^2\right)^{\frac{1}{2}}. \tag{2.3}$$

Let a' be the element obtained from a by replacing every e_i in (2.2) by $-e_i$ and a^* be the element obtained from a by reversing the order of the factors in each $E_v = e_{v_1}e_{v_2}\cdots e_{v_p}$, and $\bar{a} = (a')^* = (a^*)'$. Clearly, $(a+b)' = a'+b'$, $(ab)' = a'b'$, and $(ab)^* = b^*a^*$.

All the elements $x = x_0 + x_1e_1 + \cdots + x_{n-1}e_{n-1}$ ($x_k \in \mathrm{R}$, $k = 0,1,\ldots,n-1$) are said to be the *vectorial* elements in A_n, denoted by $x \in \mathbf{R}^n$. Let Γ_n be the set of all elements in A_n which can be expressed as a finite product of non-zero vectors in A_n. This is said to be *the n-dimensional Clifford group.*

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be *an n-dimensional Clifford matrix* if

(i) $a, b, c, d \in \Gamma_n \cup \{0\}$;

(ii) $det(A) = ad^* - bc^* = 1$;

(iii) ab^*, bd^*, ac^*, $cd^* \in \mathbf{R}^n$.

Let $SL(2, \Gamma_n)$ denote the group of all n-dimensional Clifford matrices with a matrix product operation. Set

$$PSL(2, \Gamma_n) = SL(2, \Gamma_n)/\{\pm I\},$$

where I is the unit matrix.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \Gamma_n)$ correspond to the mapping in $\bar{\mathbf{R}}^n$

$$x \mapsto Ax = (ax+b)(cx+d)^{-1}. \tag{2.4}$$

This is an isomorphic correspondence between $PSL(2, \Gamma_n)$ and $M(\bar{\mathbf{R}}^n)$, hence these are not distinguished.

Let A_{n+1} be the associative algebra generated by $1, e_1, e_2, \ldots, e_{n-1}$ and e_n which contains A_n as a subalgebra. Let $\tilde{f} \in M(\bar{\mathbf{R}}^{n+1})$ denote the Poincaré extension of $f \in M(\bar{\mathbf{R}}^n)$.[3] Write

$$fix(f) = \{x \in \bar{\mathbf{R}}^n : f(x) = x\},$$
$$fix(\tilde{f}) = \{z = x + te_n \in \mathbf{H}^{n+1} : \tilde{f}(z) = z\},$$

where $\mathbf{H}^{n+1} = \{x_0 + x_1e_1 + \cdots + x_{n-1}e_{n-1} + te_n : x_k \in \mathbf{R},\ t > 0\} \subset \mathbf{R}^{n+1}$. For $f \in M(\bar{\mathbf{R}}^n)$, we can say that

(i) f is *fixed-point-free* if $card(fix(f)) = 0$;

is *parabolic* if $card\,(fix\,(f)) = 1$ and $card\,(fix\,(\tilde{f})) = 0$;
is *loxodromic* if $card\,(fix\,(f)) = 2$ and $card\,(fix\,(\tilde{f})) = 0$;
is *elliptic* if $card\,(fix\,(f)) \neq 0$ and $card\,(fix\,(\tilde{f})) \neq 0$,
$l\,(M)$ is the number of elements in the set M.
nown [6),8)] that

ι **2.1.** *f is fixed-point-free if and only if $card\,(fix\,(\tilde{f})) = 1$, and then f is elliptic if and* $l\,(fix\,(\tilde{f})) > 1$.

ows that

.1. *Let* $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(\bar{\mathbf{R}}^n)$. *Then*

: *loxodromic if f conjugates to* $\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix}$,
here $r > 0$, $r \neq 1$, $\lambda \in \Gamma_n$ and $|\lambda| = 1$;
s *parabolic if f conjugates to* $\begin{pmatrix} \lambda & u \\ 0 & \lambda' \end{pmatrix}$,
here $\lambda, u \in \Gamma_n$, $|\lambda| = 1$, $u \neq 0$, *and* $\lambda u = u \lambda'$;
is elliptic if f conjugates to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$,
here $\lambda \in \Gamma_n$, $|\lambda| = 1$ *and* $\lambda \neq \pm 1$.

3. Representations of Möbius Transformations

noted that types of Möbius transformations are invariant under conjugate ıtion. In order to study the type of $f \in M(\bar{\mathbf{R}}^n)$, we may assume that $\infty \notin fix\,(f)$. e are two representations of f:

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ i.e. } f(x) = (ax+b)(cx+d)^{-1},\ c \neq 0 \tag{3.1}$$

$$(x) = u + \frac{r^2 A\,(x-v)}{|x-v|^2}, \tag{3.2}$$

$\in \mathbf{R}^n$, $r > 0$, A is a real orthogonal n-order matrix and $\det(A) = -1$.
ows that the isometric sphere of f is

$$_f = \{\, x \in \mathbf{R}^n \colon |x-v| = r \,\}, \tag{3.3}$$

c^{-1}, $v = -c^{-1}d$, $r = |c|^{-1}$, and

$$Ax = -\frac{c'}{|c|}x'\frac{\bar{c}}{|c|} \text{ for } ^{\forall}x \in \bar{\mathbf{R}}^n, \tag{3.4}$$

where x is of the form $x = (x_0, \ldots, x_{n-1})^T$ at the left of the equality and the form $x = x_0 + x_1 e_1 + \ldots + x_{n-1} e_{n-1}$ at the right of the equality.

This shows that A depends only on c, denoted by $A(c)$.

The Poincaré extension of f is

$$\tilde{f}(z) = \hat{u} + \frac{r^2 B(z - \hat{v})}{|z - \hat{v}|^2}, \quad z = x + te_n \in \mathbf{H}^{n+1}, \tag{3.5}$$

where $\hat{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $\hat{v} = \begin{pmatrix} v \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, or it is written as follows.

$$\tilde{f}(z) = \frac{(ax + b)\overline{(cx + d)} + t^2 ac + te_n}{|cx + d|^2 + t^2|c|^2}. \tag{3.6}$$

Proposition 3.1. *Let* $c \in \Gamma_n \setminus \{0\}$. *Then* $uc \neq -(cu)'$ *for* $^{\forall}u \in \mathbf{R}^n \setminus \{0\}$ *if and only if* $\det(I - A(c)) \neq 0$.

proof. $uc = -(cu)'$ if and only if $-c'u'c^{-1} - u = 0$, that is $A(c)u - u = 0$. And then $uc \neq -(cu)'$ if and only if $\det(I - A(c)) \neq 0$. □

Proposition 3.2.[6] *If* $\det(I - A(c)) \neq 0$, *then* f *is fixed-point-free, loxodromic or parabolic. Thus,* $\det(I - A(c)) = 0$ *if* f *is elliptic.*

4. Type Criteria

For fixed-point-free elements and elliptic elements we have the following theorem.

Theorem 4.1. *Let* $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(\bar{\mathbf{R}}^n)$, $c \neq 0$. *Then*

(*i*) *f is fixed-point-free if and only if* $\det(I - A(c)) \neq 0$ *and*

$$|[I - A(c)]^+(ac^{-1} + c^{-1}d)| < |c|^{-1}; \tag{4.1}$$

(*ii*) *f is elliptic if and only if* $\det(I - A(c)) = 0$ *and*

$$|[I - A(c)]^+(ac^{-1} + c^{-1}d)| < |c|^{-1}, \tag{4.2}$$

where M^+ is the Moore-Penrose inverse of a matrix M.

Proof. It is known from [6] that if f is fixed-point-free, then $\det(I - A(c)) \neq 0$ and the condition (4.1) holds.

Conversely, the condition $\det(I-A(c))\neq 0$ implies that f is not elliptic by Proposition 3.2. From condition (4.1) it follows that

$$z=x_0+t_0e_n\in fix(\tilde{f}), \tag{4.3}$$

where

$$x_0=[I-A(c)]^{+}(ac^{-1}+c^{-1}d)-c^{-1}d \tag{4.4}$$

and

$$t_0=\sqrt{|c|^{-2}-|[I-A(c)]^{+}(ac^{-1}+c^{-1}d)|^2}>0. \tag{4.5}$$

Therefore, f is a fixed-point-free element. (i) is proved.

It is well known from [1] that if f is elliptic, then

$$fix(f)\cap S_f\cap S_{f^{-1}}\neq\emptyset.$$

It follows by simple calculation that $\det(I-A(c))=0$ and the condition (4.2) holds.

Conversely, the condition (4.2) implies that $fix(\tilde{f})\neq\emptyset$ for the same reason as above. It shows that f is elliptic if $\det(I-A(c))=0$. (ii) is proved. □

In order to establish the type criteria theorem for parabolic elements, we can show that parabolic elements have an interesting geometric quality as follows.

Lemma 4.1. *If $f=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is parabolic with $c\neq 0$, then the isometric spheres S_f and $S_{f^{-1}}$ of f and f^{-1} are outward tangential.*

Proof. Let $\sigma=\frac{1}{2}(ac^{-1}+c^{-1}d)$, $g=\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$, where $\beta=\frac{1}{2}(c^{-1}d-ac^{-1})$. Then, $\sigma\neq 0$, $|\sigma|=|c|^{-1}$ (see [1]) and

$$gfg^{-1}=\begin{pmatrix}\sigma c & 0\\ c & c\sigma\end{pmatrix},\ gf^{-1}g^{-1}=\begin{pmatrix}c^{-1}\sigma^{-1} & 0\\ -\sigma^{-1}c^{-1}\sigma^{-1} & \sigma^{-1}c^{-1}\end{pmatrix}.$$

The isometric spheres

$$S_{gfg^{-1}}=\{x\in\bar{\mathbf{R}}^n: |x+\sigma|=|c|^{-1}\},\ S_{gf^{-1}g^{-1}}=\{x\in\bar{\mathbf{R}}^n: |x-\sigma|=|\sigma c\sigma|\}$$

are outward tangential, as are the isometric spheres S_f and $S_{f^{-1}}$.

Theorem 4.2. $f=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ *with $c\neq 0$ is parabolic if and only if*

$$|(I-A(c))^{+}(ac^{-1}+c^{-1}d)| = |c|^{-1}, \tag{4.6}$$

$$|ac^{-1}+c^{-1}d| = 2|c|^{-1} \tag{4.7}$$

and

$$[I-(I-A(c))(I-A(c))^{+}](ac^{-1}+c^{-1}d) = 0. \tag{4.8}$$

Proof. Suppose that f is parabolic. Then, the condition (4.7) holds and the equation system

$$\begin{cases} x = ac^{-1}+A(c)(x+c^{-1}d) \\ |x+c^{-1}d| = |c|^{-1} \end{cases} \tag{4.9}$$

has a unique solution.[6] Therefore, (4.6) and (4.8) follow.

Conversely, suppose that the three equalities (4.6)-(4.8) hold. Let

$$x = -c^{-1}d+(I-A(c))^{+}(ac^{-1}+c^{-1}d). \tag{4.10}$$

It follows from (4.6) and (4.8) that $x \in S_f \cap fix(f)$. Thus, f is not loxodromic. On the other hand, it is known from (4.7) that the isometric spheres S_f and $S_{f^{-1}}$ are outward tangential. This means that f is not fixed-point-free or elliptic because of (4.6). Therefore, f is parabolic. □

Following from Theorem 4.2

Corollary 4.1. *Suppose that* $\det(I-A(c)) \neq 0$. *Then f is parabolic if and only if (4.6) and (4.7) hold.*

For loxodromic elements we obtain the following result from Theorems 4.1 and 4.2.

Theorem 4.3. $f=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $c \neq 0$ *is loxodromic if and only if one of the following conditions holds :*

(*i*) $|(I-A(c))^{+}(ac^{-1}+c^{-1}d)| > |c|^{-1}$;
(*ii*) $|(I-A(c))^{+}(ac^{-1}+c^{-1}d)| = |c|^{-1}$ *and* $|ac^{-1}+c^{-1}d| \neq 2|c|^{-1}$;
(*iii*) $|(I-A(c))^{+}(ac^{-1}+c^{-1}d)| = |c|^{-1}$ *and* $[I-(I-A(c))(I-A(c))^{+}](ac^{-1}+c^{-1}d) \neq 0$.

5. A Chordal Distortion Theorem

In this section we prove a chordal distortion theorem for elements in $M(\bar{\mathbf{R}}^n)$.

Theorem 5.1. *Let K be a compact subset of a domain D in* $\bar{\mathbf{R}}^n$. *Suppose that*

$$f=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(\bar{\mathbf{R}}^n) \text{ and } g^{-1}(0),\ g^{-1}(\infty) \notin D.$$

Then

$$d(g(x), g(y)) \leq \frac{d(x, y)}{m^2 \|g\|^2}, \tag{5.1}$$

where $d(u, v)$ *is the chordal distance between* u *and* v,[3)]

$$m = \frac{1}{2} inf \{d(u, v) : u \notin D, v \in K\}. \tag{5.2}$$

Proof. Because $g^{-1}(0), g^{-1}(\infty) \notin D$, we have

$$2m \leq d(x, g^{-1}(\infty)) \leq \frac{2 \mid x + d^*(c^*)^{-1} \mid}{(1 + |x|^2)^{\frac{1}{2}}(1 + |d|^2 |c^{-1}|^2)^{\frac{1}{2}}}$$

$$= \frac{2 \mid cx + d \mid}{(1 + |x|^2)^{\frac{1}{2}}(|c|^2 + |d|^2)^{\frac{1}{2}}}$$

and

$$2m \leq \frac{2 \mid ax + b \mid}{(1 + |x|^2)^{\frac{1}{2}}(|a|^2 + |b|^2)^{\frac{1}{2}}}.$$

Then

$$(1 + |x|^2) m^2 \|g\|^2 \leq |ax + b|^2 + |cx + d|^2. \tag{5.3}$$

We know that [1)]

$$g(x) - g(y) = ((cx + d)^{-1})^* (x - y)(cy + d)^{-1}. \tag{5.4}$$

It follows that

$$\frac{d(g(x), g(y))}{d(x, y)} \leq \left(\frac{1 + |x|^2}{|ax + b|^2 + |cx + d|^2} \right)^{\frac{1}{2}} \left(\frac{1 + |y|^2}{|ay + b|^2 + |cy + d|^2} \right)^{\frac{1}{2}}. \tag{5.5}$$

Therefore, the theorem is proved. □

Department of Mathematics (X. W.),
Hunan University and Wuhan University,
Changsha, Hunan 410082, P. R. China
Department of Applied Mathematics (W. Y.),
Beijing Institute of Technology,
Beijing 100081, P. R. China
Department of Mechanical and System Engineering (F. M.),
Faculty of Engineering and Design,
Kyoto Institute of Technology,
Matsugasaki, Sakyo-ku, Kyoto 606-8585

Department of Mathematics (Y. J.),
Hunan University,
Changsha, Hunan 410082, P. R. China

References

1) L. Ahlfors, Ann. Acad. Sci. Fen. Ser. A. I. Math., **10**, 15-27 (1985).
2) L. Ahlfors, Möbius transformations and Clifford numbers., In "Differential Geometry and Complex Analysis", Springer-Verlag, Berlin, (1985).
3) A. F. Beardon, "The geometry of discrete groups", p. 337, Springer-Verlag, (1983).
4) A. Fang, *Acta Math. Sinica,* **9**, 231-239 (1993).
5) C. Liu, *Math. Advance of China,* **19**, 231-239 (1990).
6) X. Wang, *Jour. of Hunan University,* **19**, 128-135 (1992).
7) P. Waterman, *Adv. in Math.,* **101**, 87-113 (1993).
8) Z. Yu, J. Wang and F. Ren, *Jour. of Fudan University,* **35**, 374-380 (1996).