A Note on the Stability of Delta-Operator-Induced Systems

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***Abstract*—This correspondence points out that stability of delta-operator-induced real polynomials is equivalent to a shifted Hurwitz stability of their reciprocal counterparts. This fact reveals a simple and clear link between the delta-domain stability and Hurwitz stability for real polynomials.**

***Index Terms*—Delta-operator, Hurwitz stability, real polynomials, stability property.**

I. Introduction

It is well known that the delta-operator mediates between the differential and difference operators and unifies two parallel control (or signal processing) theories (continuous-time theory and sampled-data one) by taking the sampling interval as a parameter [1]. So far as stability of systems induced by the delta-operator is concerned, several test methods have been thus far proposed and discussed for polynomials or state-space models [1]–[6].

In general, as the sampling interval decreases and comes close to zero, the delta-domain stability approaches Hurwitz stability and some of the aforementioned articles touch on this subject using their own algorithms. In the authors' view, however, the connection between the two stability concepts is by no means fully clear and smooth. This correspondence contributes to clarifying this "junction" problem. It is pointed out that the delta-domain stability of real polynomials is nothing other than a shifted Hurwitz property of their reciprocal counterparts. This fact reveals a link between the two stabilities in a simple, clear, and general manner.

II. A Link Between the Delta-Domain Stability and Hurwitz Stability

Delta-operator is defined by $\delta = (q-1)/T$ with q being the shift operator and T the sampling interval. After [1], we employ γ to denote the frequency domain variable corresponding to δ and consider nth degree real polynomials,

$$f(\gamma) = a_0\gamma^n + a_1\gamma^{n-1} + \cdots + a_n, \qquad a_0 > 0. \tag{1}$$

It is known that systems with $f(\gamma)$ as the characteristic polynomials are stable (in the δ domain), if and only if the zeros of $f(\gamma)$ all satisfy $\gamma \in \mathcal{D}$ where

$$\mathcal{D} := \left\{ z = x + jy,\ x \in \Re,\ jy \in \Im \,\middle|\, \left(x + \frac{1}{T}\right)^2 + y^2 < \frac{1}{T^2} \right\}. \tag{2}$$

Here, j stands for the imaginary unit. In what follows, we say $f(\gamma)$ is $\mathcal{D}$-stable, if the above condition is fulfilled by its zeros. We assume the $\mathcal{D}$-stability of (1), which leads to $a_i > 0,\ i = 1, \ldots, n$, because $\mathcal{D}$ lies in the Hurwitz region (i.e., open left half complex plane). Now we observe that $\mathcal{D}$ is the disk region in the complex plane centered at $(-1/T, j0)$ with radius $1/T$. As T tends to 0, the disk apparently expands monotonically and finally covers the whole Hurwitz region. Accordingly, the same should occur in the polynomial coefficient space. To imagine this, consider the $\mathcal{D}$-stability region in the coefficient space with T as a parameter. The region must continually expand ,as T decreases, and reach its limit, Hurwitz stability region in the coefficient space. Our interest lies in what way this smooth transition in the coefficient space can be generally realized and convinced.

The key to this problem is to introduce the following inverse of $\mathcal{D}$-stability region in the complex plane.

$$\mathcal{D}_{-1} := \{z \in C | 1/z \in \mathcal{D}\}. \tag{3}$$

Note that since any zero of (1) does not vanish, $\mathcal{D}_{-1}$ is well-defined and $z \in \mathcal{D}$ if and only if $1/z \in \mathcal{D}_{-1}$. A little manipulation shows that $\mathcal{D}_{-1}$ is given by

$$\mathcal{D}_{-1} = \left\{ z = x + jy,\ x \in \Re,\ jy \in \Im | x < -\frac{T}{2} \right\}. \tag{4}$$

With this inverse region, the main result can be stated as follows.

Theorem 1: A necessary and sufficient condition for $f(\gamma)$ to be $\mathcal{D}$-stable is that the polynomial $g(s - T/2)$ is Hurwitz stable. Here,

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$g(s)$ is the reciprocal polynomial of $f(\gamma)$ given by $g(s) = s^n f(1/s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$.

Proof: This is an immediate consequence of the above observations. See also [1, p. 135].

The significance of this theorem lies in that it gives a link between $\mathcal{D}$-stability and Hurwitz stability of real polynomials in a simple, clear and general manner. We can realize at a glance the smooth, continual expansion or transition in the general stability conditions from $\mathcal{D}$-stability to Hurwitz one as T tends to 0 without resorting to specific $\mathcal{D}$-stability tests as attempted in [2] or [3]. It is interesting to note that the limit of $\mathcal{D}$-stability as T goes to 0 is not Hurwitz stability of $f(\gamma)$ itself, but of its reciprocal counterpart. However, both are, of course, identical. We also note that the theorem can provide a useful viewpoint in employing any of conventional Hurwitz stability tests for $\mathcal{D}$-stability checking.

Finally, we consider an extension of the previous theorem to systems formulated in the state-space. This may enable us to view the result in a little wider perspective.

Let us consider a delta-operator-induced state-space model,

$$\delta \boldsymbol{x}(t) = A\boldsymbol{x}(t), \qquad A \in R^{n \times n}. \tag{5}$$

The characteristic polynomial of this system has the form of (1) with $a_0 = 1$ and the matrix A is nonsingular if $\mathcal{D}$-stability is assured. The direct consequence of Theorem 1 for this system is the following statement.

Theorem 2: System (5) is $\mathcal{D}$-stable, if and only if the matrix $A^{-1} + (T/2)I$, where I is the identity matrix, is Hurwitz stable.

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