Memoirs of the Faculty of Engineering and Design
Kyoto Institute of Technology 46 (March, 1998)

A New Approach to the Skew Product of Symmetric Markov Processes

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(Received August 18, 1997; Accepted August 28, 1997)

Abstract

The skew product of two independent symmetric Markov processes $X_t^{(1)}$ and $X_t^{(2)}$ is defined to be the process $(X_t^{(1)}, X_{A(t)}^{(2)})$, where $A(t)$ is a positive continuous additive functional of the first process. Both explicit formula and core of the Dirichlet form of the skew product process have been determined by Fukushima-Oshima[1] for conservative symmetric diffusion processes. This formula and the regularity of the Dirichlet form for conservative symmetric Markov processes have already been established by the present author.[4] In the present paper, a simple proof of this formula will be given, along with detailed information on the core, for symmetric Markov processes which are not necessarily conservative. In the proof, Dirichlet forms perturbed by killing transformations are used, instead of the time change transformations used in previous research. Some results related to Fubini-type theorems and applications will also be given.

Key Words: *Symmetric Markov process; skew product; Dirichlet form; capacity; killing transformation; irreducibility; recurrence; transience.*

1. Introduction

Let $X^{(i)}$ $(i=1, 2)$ be two locally compact, separable metric spaces and let $X:=X^{(1)}\times X^{(2)}$. Let $\mathbf{M}^{(i)}=(\Omega^{(i)}, X_t^{(i)}, \zeta^{(i)}, P_{x^{(i)}}^{(i)})$ $(i=1, 2)$ be two Markov processes on $X^{(i)}$ $(i=1, 2)$, respectively, and let (A_t) be a positive continuous additive functional (PCAF, for short) of $\mathbf{M}^{(1)}$.

The *skew product* of $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ with respect to (A_t) is defined to be the Markov process $\mathbf{M}=(\Omega, X_t, \zeta, P_x)$ on the product space X given by

$$\Omega := \Omega^{(1)} \times \Omega^{(2)}, \tag{1.1}$$

$$X_t(\omega) := (X_t^{(1)}(\omega^{(1)}), X_{A_t(\omega^{(1)})}^{(2)}(\omega^{(2)})) \qquad (t < \zeta(\omega)), \tag{1.2}$$

$$\zeta(\omega) := \zeta^{(1)}(\omega^{(1)}) \wedge \sup\{t \geq 0;\ A_t(\omega^{(1)}) < \zeta^{(2)}(\omega^{(2)})\}, \tag{1.3}$$

$$P_x := P_{x^{(1)}}^{(1)} \otimes P_{x^{(2)}}^{(2)}, \tag{1.4}$$

where $\omega = (\omega^{(1)}, \omega^{(2)}) \in \Omega$, and $x = (x^{(1)}, x^{(2)}) \in X$.

We assume that the transition functions of $\mathbf{M}^{(i)}$ $(i=1, 2)$ are symmetric with respect to certain positive Radon measures $m^{(i)}$ $(i=1, 2)$ on $X^{(i)}$ $(i=1, 2)$, respectively, with full supports. It follows that the Dirichlet forms $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ $(i=1, 2)$ of $\mathbf{M}^{(i)}$ $(i=1, 2)$ on $L^2(X^{(i)}, m^{(i)})$ $(i=1, 2)$, respectively, are well defined. We further assume that $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ $(i=1, 2)$ are regular and that the Revuz measure μ of PCAF (A_t) is a positive Radon measure on $X^{(1)}$. Let $(\mathcal{E}^{(1),\mu}, \mathcal{F}^{(1),\mu})$ be the perturbed Dirichlet form on $L^2(X^{(1)}, m^{(1)})$ corresponding to the killing transform $\mathbf{M}^{(1),A}$ of $\mathbf{M}^{(1)}$ by PCAF (A_t) (see Section 2).

Let $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ be any cores of $(\mathcal{E}^{(1),\mu}, \mathcal{F}^{(1),\mu})$ and $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$, respectively. Since the transition function of the skew product process $\mathbf{M}$ is symmetric with respect to the product measure $m := m^{(1)} \otimes m^{(2)}$, its Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ is also well defined. Let $\mathcal{C}^{(1)} \otimes \mathcal{C}^{(2)}$ denote the linear span of the set of all functions of the form $u^{(1)}(x^{(1)})u^{(2)}(x^{(2)})$, where $u^{(i)} \in \mathcal{C}^{(i)}$ $(i=1, 2)$. In Section 3 (Theorem 3.1), we will prove that $\mathcal{C}^{(1)} \otimes \mathcal{C}^{(2)}$ is a core of $(\mathcal{E}, \mathcal{F})$, and that the following formula with $\mathcal{C} = \mathcal{C}^{(1)} \otimes \mathcal{C}^{(2)}$ holds true:

$$\begin{aligned}\mathcal{E}(u, v) = &\int_{X^{(2)}} \mathcal{E}^{(1)}(u(\cdot, x^{(2)}), v(\cdot, x^{(2)}))dm^{(2)}(x^{(2)}) \\ &+ \int_{X^{(1)}} \mathcal{E}^{(2)}(u(x^{(1)}, \cdot), v(x^{(1)}, \cdot))d\mu(x^{(1)})\end{aligned} \qquad (u, v \in \mathcal{C}). \qquad (1.5)$$

Fukushima and Oshima[1] have proved that this formula with $\mathcal{C} = C_0^\infty(X)$ holds true, and that $C_0^\infty(X)$ is a core of $(\mathcal{E}, \mathcal{F})$, where $\mathbf{M}^{(i)}$ $(i=1, 2)$ are conservative diffusion processes on smooth manifolds, and $C_0^\infty(X^{(i)})$ $(i=1, 2)$ are cores of $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ $(i=1, 2)$, respectively, under some additional assumptions. Here $C_0^\infty(X)$ and $C_0^\infty(X^{(i)})$ $(i=1, 2)$ denote the linear spaces of all infinitely differentiable functions on X and $X^{(i)}$ $(i=1, 2)$, respectively, with compact support. Since these results can be derived from Theorem 3.1 without additional assumptions, their results have been extended herein to general symmetric Markov processes which are not necessarily conservative. In a former work by the present author,[4] eq. (1.5) was proved where $\mathbf{M}^{(i)}$ $(i=1, 2)$ are conservative, without explicitly specifying core $\mathcal{C}$ of $(\mathcal{E}, \mathcal{F})$. Kuwae[3] has also shown that $\mathcal{C}^{(1)} \otimes \mathcal{C}^{(2)}$ is a core of $(\mathcal{E}, \mathcal{F})$ where $\mathbf{M}^{(i)}$ $(i=1, 2)$ are conservative and $\mathcal{C}^{(i)}$ $(i=1, 2)$ are special standard cores of $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ $(i=1, 2)$, respectively, in the sense of Fukushima-Oshima-Takeda.[2]

While Section 2 recalls preliminary facts concerning symmetric Markov processes and Dirichlet forms, our main theorem is stated and proved in Section 3. Section 4 is on Fubini-type theorems, both for functions in the Dirichlet space of the skew product process and for capacities. Section 5 reports on applications for the results given in Sections 3 and 4 and discusses global properties of symmetric Markov processes.

2. Symmetric Markov Processes and Dirichlet Forms

In this section, we recall some generalities of symmetric Markov processes and Dirichlet forms. We refer readers to the work of Fukushima-Oshima-Takeda[2)] for details.

In the following, for any bilinear form $\mathcal{E}$ and any measure μ, we use the notation:

$$\|u\|_{\mathcal{E}} :=\sqrt{\mathcal{E}(u, u)},\quad (u, v)_{\mu} := \int uv\, d\mu,\quad \|u\|_{\mu} :=\sqrt{(u, u)_{\mu}}.$$

Let X be a locally compact, separable metric space and let $\mathbf{M}:=(\Omega, X_t, \zeta, P_x)$ be a Hunt process on X. A Hunt process is a special Markov process possessing the right continuity of sample paths, the quasi-left continuity and the strong Markov property.[2)] Let p_t be the transition function of $\mathbf{M}$, *i.e.*, $p_t(x, dy) = P_x(X_t \in dy, t < \zeta)$ $(x \in X, t \geq 0)$. For any non-negative or bounded Borel function f on X, we have $p_t f(x) = E_x[f(X_t)]$ by the usual convention that $f(X_t)=0$ for any $t \geq \zeta$, where $E_x[\cdot]$ denotes the expectation with respect to P_x. Suppose that p_t is symmetric with respect to a certain positive Radon measure m on X with full support, *i.e.*, $(p_t f, g)_m=(f, p_t g)_m$ for any non-negative Borel functions f and g on X. Then p_t generates a strongly continuous semigroup $\{T_t\}$ of symmetric Markovian operators on the real Hilbert space $L^2(X, m)$ of all real square-integrable functions on X with respect to m. The *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ of $\mathbf{M}$ is defined by

$$\mathcal{F} := \{u \in L^2(X, m);\ \sup_{t>0} \frac{1}{t}(u - T_t u, u)_m < \infty\}, \tag{2.1}$$

$$\mathcal{E}(u, v) := \lim_{t \downarrow 0} \frac{1}{t}(u - T_t u, v)_m \qquad (u, v \in \mathcal{F}). \tag{2.2}$$

Let $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)_m$ for any $u, v \in \mathcal{F}$ and any $\alpha > 0$. It is well known that $(\mathcal{F}, \mathcal{E}_1)$ is a real Hilbert space having $\mathcal{E}_1(\cdot, \cdot)$ as its inner product. We denote by $\mathcal{F}_e$ the family of all m-measurable functions u on X, admitting an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of elements in $\mathcal{F}$, such that $\lim_{n\to\infty} u_n=u$ exists and finite m-a.e. It is known that $\mathcal{F}_e$ is a linear space containing $\mathcal{F}$ and to which the bilinear form $\mathcal{E}$ extends. We call $(\mathcal{F}_e, \mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{F})$.

Let L be the infinitesimal generator of $\{T_t\}$. By the spectral representation theorem for self-adjoint operators, there exists a Borel measure $E(\cdot)$ on $[0, \infty)$ whose values are orthogonal projections on $L^2(X, m)$, such that

$$-L = \int_0^\infty \lambda\, dE(\lambda), \tag{2.3}$$

$$T_t = \int_0^\infty e^{-\lambda t} dE(\lambda), \tag{2.4}$$

$$\mathcal{E}(u, v) = \int_0^\infty \lambda\, d(E(\lambda)u, v)_m. \tag{2.5}$$

We call $E(\cdot)$ the *spectral measure* corresponding to $\{T_t\}$. Using these relations we obtain a lemma needed later:

Lemma 2.1. *Let* $\{T_t\}$ *and* $(\mathscr{E}, \mathscr{F})$ *be as above.*

(1) *If* $u \in L^2(X, m)$ *and* $t > 0$, *then* $T_t u \in \mathscr{F}$ *and* $\| T_t u \|_{\mathscr{E}} \leq \| u \|_m / \sqrt{2te}$.

(2) *If* $u \in \mathscr{F}$, *then* $T_t u \underset{(t \downarrow 0)}{\longrightarrow} u$ *in* $(\mathscr{F}, \mathscr{E}_1)$.

(3) *If* $\mathscr{D}$ *is a dense subset of* $L^2(X, m)$, *then* $\cup_{t>0} T_t \mathscr{D}$ *is dense in* $(\mathscr{F}, \mathscr{E}_1)$.

We omit the proof since it is elementary.

Let $C_0(X)$ be the linear space of all real continuous functions on X with compact support. A subset $\mathscr{C}$ of $\mathscr{F} \cap C_0(X)$ is said to be a *core* of $(\mathscr{E}, \mathscr{F})$ if it is dense in $(\mathscr{F}, \mathscr{E}_1)$ and uniformly dense in $C_0(X)$. A Dirichlet form is said to be *regular* if it possesses a core. It is known[2)] that for any regular symmetric Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(X, m)$ there exists an m-symmetric Hunt process on X whose Dirichlet form is $(\mathscr{E}, \mathscr{F})$.

The *capacity* relative to $\mathscr{E}$, or $\mathscr{E}$*-capacity*, is the set function Cap$(\cdot)$ on X defined by

$$\mathrm{Cap}(G) := \inf\{\mathscr{E}_1(u, u);\ u \in \mathscr{F},\ u \geq 1\ \ m\text{-a.e. on } G\} \qquad \text{for any open set } G, \tag{2.6}$$

$$\mathrm{Cap}(B) := \inf\{\mathrm{Cap}(G);\ G \text{ is open and } G \supset B\} \qquad \text{for any set } B. \tag{2.7}$$

In the following $\mathscr{E}$-*q.e.* abbreviates "$\mathscr{E}$-quasi-everywhere", which means "except on a set of zero $\mathscr{E}$-capacity". A function f defined $\mathscr{E}$-q.e. on X is said to be $\mathscr{E}$*-quasi-continuous* if for any $\varepsilon > 0$ there exists an open subset G of X such that $\mathrm{Cap}(G) < \varepsilon$ and the restriction of f on $X \setminus G$ is continuous. A function f is said to be an $\mathscr{E}$*-quasi-continuous m-version* of g if f is $\mathscr{E}$-quasi-continuous and $f = g$ m-a.e. If $\mathscr{E}$ is regular, then any $u \in \mathscr{F}_e$ admits an $\mathscr{E}$-quasi-continuous m-version, which will be denoted by $\tilde{u}$ or $(u)^{\sim}$.

Suppose that $\mathscr{E}$ is regular. Let (A_t) be a PCAF, admitting exceptional sets, in the sense of Fukushima-Oshima-Takeda[2)]. It follows that there exists a unique positive Borel measure μ on X such that

$$\lim_{t \downarrow 0} \frac{1}{t} \int_X E_x [\int_0^t f(X_s) dA_s] dm(x) = \int_X f(x) d\mu(x) \tag{2.8}$$

for any non-negative Borel function f on X. The measure μ is called the *Revuz measure* of (A_t). It is known[2)] that eq. (2.8) gives a one-to-one correspondence between the family of all equivalence classes of PCAF's of $\mathbf{M}$ and a certain family S of positive Borel measures on X, charging no set of zero capacity. Each element of S is called a *smooth* measure[2)] on X. It is known[2)] that any positive Radon measure, charging no set of zero capacity, is a smooth measure on X. Suppose that the Revuz measure μ of (A_t) is a Radon measure on X. It follows that we can define a bilinear form $(\mathscr{E}^{\mu}, \mathscr{F}^{\mu})$ by

$$\mathscr{F}^{\mu} := \{u \in \mathscr{F};\ \|\tilde{u}\|_{\mu} < \infty\}, \tag{2.9}$$

$$\mathscr{E}^{\mu}(u, v) := \mathscr{E}(u, v) + (\tilde{u}, \tilde{v})_{\mu} \qquad (u, v \in \mathscr{F}^{\mu}). \tag{2.10}$$

It is known[2] that $(\mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ is the Dirichlet form on $L^2(X, m)$, corresponding to the m-symmetric Markov process $\mathbf{M}^A$, whose transition function p_t^A is given by

$$p_t^A f(x) := E_x[e^{-A_t} f(X_t)] \quad \text{for any bounded Borel function } f \text{ on } X. \tag{2.11}$$

We call $\mathbf{M}^A$ the *killing transform* of $\mathbf{M}$ by PCAF (A_t).

For any $\alpha > 0$, an element u in $L^2(X, m)$ is said to be α*-excessive* if $u \geq 0$ and $e^{-\alpha t} T_t u \leq u$ m-a.e. for any $t > 0$. The following lemma[2] will be used in Section 4:

Lemma 2.2. *Suppose that* $u \in \mathcal{F}$ *and* $\alpha > 0$. *Then* u *is* α*-excessive if and only if* $\mathcal{E}_\alpha(u, f) \geq 0$ *for any non-negative element* $f \in \mathcal{F} \cap C_0(X)$.

Finally, we recall some global properties of $\mathbf{M}$, or of $(\mathcal{E}, \mathcal{F})$. For any set B, we denote by 1_B its indicator function. A subset B of X is said to be $\mathcal{E}$*-invariant*, or $\{T_t\}$*-invariant* if $T_t(u 1_B) = (T_t u) 1_B$ for any $t > 0$ and any $u \in L^2(X, m)$. The following lemma[5] will be used in Section 5:

Lemma 2.3. *Let* $\{\chi_n\}$ *be any non-decreasing sequence of functions in* $\mathcal{F} \cap C_0(X)$ *such that* $0 \leq \chi_n \leq 1$ $(n = 1, 2, \ldots)$ *and* $\cup_n \{x;\ \chi_n(x) = 1\} = X$.

(1) *B is $\mathcal{E}$-invariant if and only if* $\chi_n 1_B \in \mathcal{F}$ *and* $\mathcal{E}(\chi_n 1_B, \chi_n 1_{B^c}) = 0$ *for all* n.

(2) *For each* n, *if* $\chi_n 1_B \in \mathcal{F}$, *then* $\mathcal{E}(\chi_n 1_B, \chi_n 1_{B^c}) \leq 0$.

Dirichlet form $\mathcal{E}$ is said to be *irreducible* if any $\mathcal{E}$-invariant set B is m-trivial, *i.e.*, $m(B) = 0$ or $m(B^c) = 0$.

For any non-negative m-integrable function f, we define the function Gf by

$$Gf := \lim_{N \to \infty} \int_0^N T_t f\, dt \in [0, \infty] \qquad m\text{-a.e.} \tag{2.12}$$

Dirichlet form $\mathcal{E}$ is said to be *transient* if $Gf < \infty$ m-a.e. for any such f, and *recurrent* if $Gf = 0$ or ∞ m-a.e. for any such f. The following facts are well established:[2]

Lemma 2.4. (1) *If* $\mathcal{E}$ *is irreducible, then* $\mathcal{E}$ *is either transient or recurrent.*
(2) $\mathcal{E}$ *is transient if and only if, whenever* $u \in \mathcal{F}_e$, $\|u\|_{\mathcal{E}} = 0$ *implies* $u = 0$ *m-a.e.*
(3) $\mathcal{E}$ *is recurrent if and only if* $1 \in \mathcal{F}_e$ *and* $\|1\|_{\mathcal{E}} = 0$.
(4) *Suppose that* $\mathcal{E}$ *is irreducible and recurrent. If* $u \in \mathcal{F}_e$ *and* $\|u\|_{\mathcal{E}} = 0$, *then there exists a constant* c *such that* $u = c$ *m-a.e.*

3. Main Theorem

We follow the notation given in Section 1. For any subsets $\mathcal{D}^{(i)}$ $(i=1, 2)$ of functions on $X^{(i)}$ $(i=1, 2)$, respectively, we denote by $\mathcal{D}^{(1)} \otimes \mathcal{D}^{(2)}$ the totality of all linear combinations of $u^{(1)} \otimes u^{(2)}$ with $u^{(i)} \in \mathcal{D}^{(i)}$ $(i=1, 2)$, where $(u^{(1)} \otimes u^{(2)})(x) := u^{(1)}(x^{(1)}) u^{(2)}(x^{(2)})$, $x = (x^{(1)}, x^{(2)})$.

The following is the main theorem:

Theorem 3.1. *If $\mathscr{C}^{(1)}$ and $\mathscr{C}^{(2)}$ are any cores of $\mathscr{E}^{(1),\mu}$ and $\mathscr{E}^{(2)}$, respectively, then $\mathscr{C}^{(1)}\otimes\mathscr{C}^{(2)}$ is a core of $\mathscr{E}$. Moreover, eq.* (1.5) *holds true for u, $v\in\mathscr{C}^{(1)}\otimes\mathscr{C}^{(2)}$. In particular, if $u^{(i)}$, $v^{(i)}\in\mathscr{C}^{(i)}$ $(i=1, 2)$, then it follows that*

$$\begin{aligned}&\mathscr{E}(u^{(1)}\otimes u^{(2)}, v^{(1)}\otimes v^{(2)}) \\ &\quad = \mathscr{E}^{(1)}(u^{(1)}, v^{(1)})(u^{(2)}, v^{(2)})_{m^{(2)}}+(u^{(1)}, v^{(1)})_{\mu}\,\mathscr{E}^{(2)}(u^{(2)}, v^{(2)}).\end{aligned}\tag{3.1}$$

The corresponding proof will be given at the end of this section.

For any $\lambda>0$, let $p_t^{(1),\lambda A}$ be the transition function of the killing transform $\mathbf{M}^{(1),\lambda A}$ of $\mathbf{M}^{(1)}$ by PCAF (λA_t):

$$p_t^{(1),\lambda A} f(\xi):=E_\xi^{(1)}[e^{-\lambda A_t} f(X_t^{(1)})],\tag{3.2}$$

where $E_\xi^{(1)}[\cdot]$ denotes the expectation with respect to $P_\xi^{(1)}$. The corresponding semigroup and Dirichlet form on $L^2(X^{(1)}, m^{(1)})$ are denoted by $\{T_t^{(1),\lambda\mu}\}$ and $(\mathscr{E}^{(1),\lambda\mu}, \mathscr{F}^{(1),\lambda\mu})$, respectively. Note that $\mathscr{F}^{(1),\lambda\mu}=\mathscr{F}^{(1),\mu}$ whenever $\lambda>0$. Let $p_t^{(2)}$ and p_t be the transition functions of $\mathbf{M}^{(2)}$ and the skew product process $\mathbf{M}$, and let $\{T_t^{(2)}\}$ and $\{T_t\}$ be the semigroups on $L^2(X^{(2)}, m^{(2)})$ and $L^2(X, m)$, generated by $p_t^{(2)}$ and p_t, respectively. Let $E^{(2)}(\cdot)$ denote the spectral measure corresponding to $\{T_t^{(2)}\}$.

The following is a key lemma:

Lemma 3.1. *If $u^{(i)}$, $v^{(i)}\in L^2(X^{(i)}, m^{(i)})$ $(i=1, 2)$, then it holds that*

$$\begin{aligned}&(T_t(u^{(1)}\otimes u^{(2)}), v^{(1)}\otimes v^{(2)})_m \\ &\quad = \int_0^\infty (T_t^{(1),\lambda\mu}\, u^{(1)}, v^{(1)})_{m^{(1)}} d(E^{(2)}(\lambda)u^{(2)}, v^{(2)})_{m^{(2)}}.\end{aligned}\tag{3.3}$$

Proof. Taking Borel versions of $u^{(i)}$ $(i=1, 2)$, we have

$$(u^{(1)}\otimes u^{(2)})(X_t)=u^{(1)}(X_t^{(1)})u^{(2)}(X_{A_t}^{(2)}),\qquad t\in[0, \infty)\tag{3.4}$$

by the usual convention, and hence,

$$p_t(u^{(1)}\otimes u^{(2)})(\xi, \eta)=E_\xi^{(1)}[u^{(1)}(X_t^{(1)})p_{A_t}^{(2)}u^{(2)}(\eta)],\qquad t\in[0, \infty).\tag{3.5}$$

Thus, using Fubini's theorem, we obtain

$$\begin{aligned}&(T_t(u^{(1)}\otimes u^{(2)}), v^{(1)}\otimes v^{(2)})_m \\ &= \int_{X^{(1)}} E_\xi^{(1)}[u^{(1)}(X_t^{(1)})(T_{A_t}^{(2)} u^{(2)}, v^{(2)})_{m^{(2)}}]v^{(1)}(\xi)dm^{(1)}(\xi) \\ &= \int_{X^{(1)}} E_\xi^{(1)}[u^{(1)}(X_t^{(1)}) \int_0^\infty e^{-\lambda A_t} d(E^{(2)}(\lambda)u^{(2)}, v^{(2)})_{m^{(2)}}]\, v^{(1)}(\xi)dm^{(1)}(\xi) \\ &= \int_0^\infty \left(\int_{X^{(1)}} E_\xi^{(1)}[u^{(1)}(X_t^{(1)})e^{-\lambda A_t}]v^{(1)}(\xi)dm^{(1)}(\xi)\right)d(E^{(2)}(\lambda)u^{(2)}, v^{(2)})_{m^{(2)}} \\ &= \int_0^\infty (T_t^{(1),\lambda\mu} u^{(1)}, v^{(1)})_{m^{(1)}}\, d(E^{(2)}(\lambda)u^{(2)}, v^{(2)})_{m^{(2)}}.\end{aligned}$$

□

Remark 3.1. While Lemma 3.1 signifies that

$$T_t(u^{(1)} \otimes u^{(2)}) = \int_0^\infty T_t^{(1),\lambda\mu} u^{(1)} \otimes dE^{(2)}(\lambda) u^{(2)} \tag{3.6}$$

holds in the weak sense of $L^2(X, m)$, this equation actually holds in the strong sense of $L^2(X, m)$.

Lemma 3.2. *If $u \in \mathcal{F}^{(1),\mu}$ and $v \in \mathcal{F}^{(2)}$, then $u \otimes v \in \mathcal{F}$ and it holds that*

$$\|u \otimes v\|_{\mathcal{E}}^2 = \|u\|_{\mathcal{E}^{(1)}}^2 \|v\|_{m^{(2)}}^2 + \|\tilde{u}\|_{\mu}^2 \ \|u\|_{\mathcal{E}^{(2)}}^2, \tag{3.7}$$

$$\|u \otimes v\|_{\mathcal{E}_1}^2 = \|u\|_{\mathcal{E}_1^{(1)}}^2 \|v\|_{m^{(2)}}^2 + \|\tilde{u}\|_{\mu}^2 \|u\|_{\mathcal{E}^{(2)}}^2 \leq \|u\|_{\mathcal{E}_1^{(1),\mu}}^2 \ \|u\|_{\mathcal{E}_1^{(2)}}^2 . \tag{3.8}$$

Furthermore, eq. (3.1) *holds for any $u^{(1)}, v^{(1)} \in \mathcal{F}^{(1),\mu}$ and any $u^{(2)}, v^{(2)} \in \mathcal{F}^{(2)}$.*

Proof. Using Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{t}(u \otimes v - T_t(u \otimes v), u \otimes v)_m \\
&= \int_0^\infty \frac{1}{t}(u - T_t^{(1),\lambda\mu} u, u)_{m^{(1)}} \, d(E^{(2)}(\lambda)v, v)_{m^{(2)}} \\
&\xrightarrow[(t \downarrow 0)]{} \int_0^\infty \mathcal{E}^{(1),\lambda\mu}(u, u) \, d(E^{(2)}(\lambda)v, v)_{m^{(2)}} \\
&= \int_0^\infty \{\mathcal{E}^{(1)}(u, u) + \lambda(\tilde{u}, \tilde{u})_\mu\} \, d(E^{(2)}(\lambda)v, v)_{m^{(2)}} \\
&= \mathcal{E}^{(1)}(u, u)(v, v)_{m^{(2)}} + (\tilde{u}, \tilde{u})_\mu \, \mathcal{E}^{(2)}(v, v).
\end{aligned}$$

This proves eq. (3.7), from which eq. (3.8) follows. The last assertion can be similarly proved. □

Lemma 3.3. *For any $u^{(i)} \in L^2(X^{(i)}, m^{(i)})$ $(i=1, 2)$ and any $t > 0$, $T_t(u^{(1)} \otimes u^{(2)})$ belongs to the closure of $\mathcal{F}^{(1),\mu} \otimes \mathcal{F}^{(2)}$ in the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$.*

Proof. For any $u^{(i)} \in L^2(X^{(i)}, m^{(i)})$ $(i=1, 2)$ and any $t > 0$ we take an approximating sequence $\{v_n\}$ for $T_t(u^{(1)} \otimes u^{(2)})$, defined by

$$v_n := \sum_{j=1}^{n^2} T_t^{(1),\frac{j}{n}\mu} u^{(1)} \otimes E^{(2)}(I_j^{[n]}) \, u^{(2)}, \tag{3.9}$$

where $I_j^{[n]} = (\frac{j-1}{n}, \frac{j}{n}]$ $(n, j = 1, 2, \ldots)$. It follows from Lemma 3.1 that the sequence $\{v_n\}$ converges to $T_t(u^{(1)} \otimes u^{(2)})$ weakly in $L^2(X, m)$. Note that $E^{(2)}(I_j^{[n]})u^{(2)} \in \mathcal{F}^{(2)}$ and, moreover,

$$\mathcal{E}^{(2)}(E^{(2)}(I_i^{[n]}) \, u^{(2)}, E^{(2)}(I_j^{[n]}) \, u^{(2)}) = 0 \qquad \text{if } i \neq j \tag{3.10}$$

and

$$\|E^{(2)}(I_j^{[n]}) \, u^{(2)}\|_{\mathcal{E}^{(2)}}^2 \leq \frac{j}{n} \|E^{(2)}(I_j^{[n]}) u^{(2)}\|_{m^{(2)}}^2 = \frac{j}{n} (E^{(2)}(I_j^{[n]}) u^{(2)}, u^{(2)})_{m^{(2)}} \tag{3.11}$$

(n, i, $j=1, 2,\dots$). Thus, using Lemma 3.2 and Lemma 2.1 (1), we obtain

$$\begin{aligned}\|v_n\|_{\mathscr{E}}^2 &= \sum_{i,j=1}^{n^2} \mathscr{E}(T_t^{(1),\frac{i}{n}\mu}u^{(1)}\otimes E^{(2)}(I_i^{[n]})u^{(2)},\ T_t^{(1),\frac{j}{n}\mu}u^{(1)}\otimes E^{(2)}(I_j^{[n]})u^{(2)})\\ &\le \sum_{i=1}^{n^2}\{\|T_t^{(1),\frac{i}{n}\mu}u^{(1)}\|_{\mathscr{E}^{(1)}}^2+\frac{i}{n}\|(T_t^{(1),\frac{i}{n}\mu}u^{(1)})^{\sim}\|_{\mu}^2\}\ (E^{(2)}(I_i^{[n]})u^{(2)},u^{(2)})_{m^{(2)}}\\ &= \sum_{i=1}^{n^2}\|T_t^{(1),\frac{i}{n}\mu}u^{(1)}\|_{\mathscr{E}^{(1),\frac{i}{n}\mu}}^2\ (E^{(2)}(I_i^{[n]})u^{(2)},u^{(2)})_{m^{(2)}}\\ &\le \frac{1}{2et}\ \|u^{(1)}\|_{m^{(1)}}^2\ \|u^{(2)}\|_{m^{(2)}}^2<\infty.\end{aligned}$$

Therefore the sequence $\{v_n\}$ is bounded in $(\mathscr{F}, \mathscr{E}_1)$. Hence, using a standard argument, we can choose a sub-sequence $\{v_{n_k}\}$ of $\{v_n\}$, such that its Cesàro means $w_n := \frac{1}{n}\sum_{k=1}^n v_{n_k}$ ($n=1, 2,\dots$) converge to an element $w^* \in \mathscr{F}$ strongly in $(\mathscr{F}, \mathscr{E}_1)$, and hence, weakly in $L^2(X, m)$. On the other hand, $\{w_n\}$ converges to $T_t(u^{(1)}\otimes u^{(2)})$ weakly in $L^2(X, m)$, concluding that $w^* = T_t(u^{(1)}\otimes u^{(2)})$. This completes the proof since $w_n \in \mathscr{F}^{(1),\mu}\otimes\mathscr{F}^{(2)}$ ($n=1, 2,\dots$). □

Now we shall give the proof of our main theorem.

Proof of Theorem 3.1. First note that eq. (1.5) follows from eq. (3.1). Given the assumptions and Lemma 3.2, it suffices to show that $\mathscr{F}^{(1),\mu}\otimes\mathscr{F}^{(2)}$ is dense in $(\mathscr{F}, \mathscr{E}_1)$. To this end we set $\mathscr{D} := L^2(X^{(1)}, m^{(1)})\otimes L^2(X^{(2)}, m^{(2)})$. It follows from Lemma 2.1 (3) that $\cup_{t>0}T_t\mathscr{D}$ is dense in $(\mathscr{F}, \mathscr{E}_1)$ since $\mathscr{D}$ is dense in $L^2(X, m)$. Therefore, it follows from Lemma 3.3 that $\mathscr{F}^{(1),\mu}\otimes\mathscr{F}^{(2)}$ is dense in $(\mathscr{F}, \mathscr{E}_1)$. □

4. Fubini-Type Results

We follow the notation used in the previous section. For any subset B of X and any point $(\xi, \eta)\in X$, we set $B_\xi := \{x^{(2)}\in X^{(2)}; (\xi, x^{(2)})\in B\}$ (*ξ-section*) and $B^\eta := \{x^{(1)}\in X^{(1)}; (x^{(1)}, \eta)\in B\}$ (*η-section*). Let $\mathrm{Cap}^{(i)}(\cdot)$ ($i=1, 2$), $\mathrm{Cap}^{(1),\mu}(\cdot)$ and $\mathrm{Cap}(\cdot)$ denote the capacities relative to $\mathscr{E}^{(i)}$ ($i=1, 2$), $\mathscr{E}^{(1),\mu}$ and $\mathscr{E}$, respectively.

Theorem 4.1. (1) *If* $u\in\mathscr{F}_e$, *then* $u(\cdot,\eta)\in\mathscr{F}_e^{(1)}$ *for* $m^{(2)}$-*a.a.* η *and*

$$\|u\|_{\mathscr{E}}^2 \ge \int_{X^{(2)}}\|u(\cdot,\eta)\|_{\mathscr{E}^{(1)}}^2\,dm^{(2)}(\eta). \tag{4.1}$$

(2) *If* $u\in\mathscr{F}$, *then* $u(\cdot,\eta)\in\mathscr{F}^{(1)}$ *for* $m^{(2)}$-*a.a.* η *and*

$$\|u\|_{\mathscr{E}_1}^2 \ge \int_{X^{(2)}}\|u(\cdot,\eta)\|_{\mathscr{E}_1^{(1)}}^2\,dm^{(2)}(\eta). \tag{4.2}$$

(3) *If* $\{B_k\}$ *is a decreasing sequence of open subsets of* X *satisfying* $\mathrm{Cap}(B_k)\downarrow 0$, *then* $\mathrm{Cap}^{(1)}(B_k^\eta)\downarrow 0$ *for* $m^{(2)}$-*a.a.* η.

(4) *If* $N\subset X$ *and* $\mathrm{Cap}(N)=0$, *then* $\mathrm{Cap}^{(1)}(N^\eta)=0$ *for* $m^{(2)}$-*a.a.* η. *In particular,* $\mu\otimes m^{(2)}(N)=0$.

(5) *If a function f is $\mathscr{E}$-quasi-continuous on X, then $f(\cdot, \eta)$ are $\mathscr{E}^{(1)}$-quasi-continuous on $X^{(1)}$ for $m^{(2)}$-a.a. η.*

Proof. Let $\mathscr{C} := \mathscr{C}^{(1)} \otimes \mathscr{C}^{(2)}$. We first note that inequalities (4.1) and (4.2) hold for any $u \in \mathscr{C}$ by Lemma 3.2. To prove (1) we take a $u \in \mathscr{F}_e$. We can choose a sequence $\{\phi_n\}$ from $\mathscr{C}$ such that $\phi_n \underset{(n\to\infty)}{\longrightarrow} u$ m-a.e. and $\sum_{n=1}^{\infty} \| \phi_{n+1} - \phi_n \|_{\mathscr{E}} < \infty$. Thus, by using inequality (4.1), we obtain

$$\left\{ \int_{X^{(2)}} \left(\sum_{n=1}^{\infty} \| \phi_{n+1}(\cdot, \eta) - \phi_n(\cdot, \eta) \|_{\mathscr{E}^{(1)}} \right)^2 dm^{(2)}(\eta) \right\}^{1/2} \leq \sum_{n=1}^{\infty} \| \phi_{n+1} - \phi_n \|_{\mathscr{E}} < \infty.$$

It follows that, for $m^{(2)}$-a.a. η, $\{\phi_n(\cdot, \eta)\}$ are $\mathscr{E}^{(1)}$-Cauchy sequences in $\mathscr{F}^{(1)}$ and converge to $u(\cdot, \eta)$ $m^{(1)}$-a.e. by Fubini's theorem. This completes the proof of (1). Assertion (2) follows from (1), Fubini's theorem and the fact that $\mathscr{F}_e^{(1)} \cap L^2(X^{(1)}, m^{(1)}) = \mathscr{F}^{(1)}$. To show (3) we take the 1-equilibrium potential[2] e_k of B_k, which satisfies

$$\mathrm{Cap}(B_k) = \| e_k \|_{\mathscr{E}_1}^2, \qquad e_k \in \mathscr{F}, \qquad e_k = 1 \ m\text{-a.e. on } B_k. \tag{4.3}$$

It follows from (2) and Fubini's theorem that

$$\| e_k(\cdot, \eta) \|_{\mathscr{E}_1^{(1)}}^2 \geq \mathrm{Cap}^{(1)}(B_k^{\eta}) \quad \text{for } m^{(2)}\text{-a.a. } \eta. \tag{4.4}$$

On the other hand, we have, by using relations (4.2) and (4.3), and Fatou's lemma,

$$\int_{X^{(2)}} \liminf_{k\to\infty} \| e_k(\cdot, \eta) \|_{\mathscr{E}_1^{(1)}}^2 \, dm^{(2)}(\eta) \leq \lim_{k\to\infty} \mathrm{Cap}(B_k) = 0. \tag{4.5}$$

Hence, $\lim_{k\to\infty} \mathrm{Cap}^{(1)}(B_k^{\eta}) = \liminf_{k\to\infty} \| e_k(\cdot, \eta) \|_{\mathscr{E}_1^{(1)}}^2 = 0$ for $m^{(2)}$-a.a. η. Assertions (4) and (5) follow from (3). □

In the following, for any $u \in \mathscr{F}_e$, an $\mathscr{E}$-quasi-continuous m-version of u will be denoted by $\tilde{u}$.

Corollary 4.1. *If $u \in \mathscr{F}_e$, then $\tilde{u}(\cdot, \eta)$ are $\mathscr{E}^{(1)}$-quasi-continuous $m^{(1)}$-versions of $u(\cdot, \eta) \in \mathscr{F}_e$ for* $m^{(2)}$-a.a. η.

This is evident from (1) and (5) of Theorem 4.1 and Fubini's theorem.

Let $\nu := \mu \otimes m^{(2)}$. It follows from Theorem 4.1 (4) that ν is a smooth Radon measure for $\mathscr{E}$. Thus we can define the perturbed Dirichlet form $(\mathscr{E}^{\nu}, \mathscr{F}^{\nu})$ corresponding to the killing transform of **M** by the PCAF whose Revuz measure is ν. We denote the capacity relative to $\mathscr{E}^{\nu}$ by $\mathrm{Cap}^{\nu}(\cdot)$.

Theorem 4.2. (1) *If $u \in \mathscr{F}_e$, then $u(\cdot, \eta) \in \mathscr{F}_e^{(1)}$ for $m^{(2)}$-a.a. η and $\tilde{u}(\xi, \cdot) \in \mathscr{F}_e^{(2)}$ for μ-a.a. ξ. Moreover, it holds that*

$$\| u \|_{\mathscr{E}}^2 = \int_{X^{(2)}} \| u(\cdot, \eta) \|_{\mathscr{E}^{(1)}}^2 dm^{(2)}(\eta) + \int_{X^{(1)}} \| \tilde{u}(\xi, \cdot) \|_{\mathscr{E}^{(2)}}^2 d\mu(\xi). \tag{4.6}$$

(2) *If* $u \in \mathscr{F}^{\nu}$, *then* $u(\cdot,\eta) \in \mathscr{F}^{(1),\mu}$ *for* $m^{(2)}$*-a.a.* η *and* $\tilde{u}(\xi,\cdot) \in \mathscr{F}^{(2)}$ *for* μ*-a.a.* ξ. *Moreover, it holds that*

$$\|u\|^2_{\mathscr{E}_1^{\nu}} = \int_{X^{(2)}} \|u(\cdot,\eta)\|^2_{\mathscr{E}_1^{(1)}}\, dm^{(2)}(\eta) + \int_{X^{(1)}} \|\tilde{u}(\xi,\cdot)\|^2_{\mathscr{E}_1^{(2)}}\, d\mu(\xi) \tag{4.7}$$

$$= \int_{X^{(2)}} \|u(\cdot,\eta)\|^2_{\mathscr{E}_1^{(1),\mu}}\, dm^{(2)}(\eta) + \int_{X^{(1)}} \|\tilde{u}(\xi,\cdot)\|^2_{\mathscr{E}^{(2)}}\, d\mu(\xi). \tag{4.8}$$

In particular, if $u \in \mathscr{F} \cap C_0(X)$, *then* $u(\cdot,\eta) \in \mathscr{F}^{(1),\mu} \cap C_0(X^{(1)})$ *for* $m^{(2)}$*-a.a.* η *and* $u(\xi,\cdot) \in \mathscr{F}^{(2)} \cap C_0(X^{(2)})$ *for* μ*-a.a.* ξ.

(3) *If* $\{B_k\}$ *is a decreasing sequence of relatively compact open subsets of* X *satisfying* $\mathrm{Cap}(B_k) \downarrow 0$, *then* $\mathrm{Cap}^{(2)}((B_k)_\xi) \downarrow 0$ *for* μ*-a.a.* ξ.

(4) *If* $N \subset X$ *and* $\mathrm{Cap}(N) = 0$, *then* $\mathrm{Cap}^{(2)}(N_\xi) = 0$ *for* μ*-a.a.* ξ.

(5) *If a function* f *is* $\mathscr{E}$*-quasi-continuous on* X, *then* $f(\xi,\cdot)$ *are* $\mathscr{E}^{(2)}$*-quasi-continuous on* $X^{(2)}$ *for* μ*-a.a.* ξ.

Proof. Let $\mathscr{C} := \mathscr{C}^{(1)} \otimes \mathscr{C}^{(2)}$. It follows from Lemma 3.2 that eqs. (4.6)–(4.8) hold for any $u \in \mathscr{C}$. To prove (1) we take a $u \in \mathscr{F}_e$ and a sequence $\{\phi_n\}$ from $\mathscr{C}$ such that $\phi_n \xrightarrow[(n\to\infty)]{} \tilde{u}$ $\mathscr{E}$-q.e. and $\sum_{n=1}^{\infty} \|\phi_{n+1} - \phi_n\|_{\mathscr{E}} < \infty$. Using the same argument as in the proof of the previous theorem and assertion (4) of the same theorem, we can show that, for μ-a.a. ξ, $\{\phi_n(\xi,\cdot)\}$ are $\mathscr{E}^{(2)}$-Cauchy sequences in $\mathscr{F}^{(2)}$ and converge to $\tilde{u}(\xi,\cdot)$ $m^{(2)}$-a.e. This proves (1) since eq. (4.6) naturally extends to $\mathscr{F}_e$. Assertion (2) follows from (1) and Fubini's theorem. To prove (3) we have only to note that $\mathrm{Cap}(B_k) \downarrow 0$ implies $\mathrm{Cap}^{\nu}(B_k) \downarrow 0$.[2)] The rest is the same as in the proof of the previous theorem. We omit the details. □

The following is a corollary to both Theorems 4.1 and 4.2:

Corollary 4.2. *Let* $B^{(i)} \subset X^{(i)}$ $(i = 1, 2)$ *and suppose that* $\mathrm{Cap}^{(i)}(B^{(i)}) > 0$ $(i = 1, 2)$. *If either* $\mu(B^{(1)}) > 0$ *or* $m^{(2)}(B^{(2)}) > 0$, *then* $\mathrm{Cap}(B^{(1)} \times B^{(2)}) > 0$.

Theorem 4.3. (1) *If* $u \in \mathscr{F}^{(1),\mu}$ *and* $v \in \mathscr{F}^{(2)}$, *then* $u \otimes v \in \mathscr{F}^{\nu}$ *and it holds that*

$$\|u \otimes v\|^2_{\mathscr{E}_1^{\nu}} = \|u\|^2_{\mathscr{E}_1^{(1)}} \|v\|^2_{m^{(2)}} + \|\tilde{u}\|^2_{\mu} \|v\|^2_{\mathscr{E}_1^{(2)}} \le \|u\|^2_{\mathscr{E}_1^{(1),\mu}} \|v\|^2_{\mathscr{E}_1^{(2)}} \tag{4.9}$$

(2) *For any* $B^{(i)} \subset X^{(i)}$ $(i=1, 2)$, $\mathrm{Cap}^{\nu}(B^{(1)} \times B^{(2)}) \le \mathrm{Cap}^{(1),\mu}(B^{(1)})\, \mathrm{Cap}^{(2)}(B^{(2)})$.

(3) *Let* $B^{(i)} \subset X^{(i)}$ $(i=1, 2)$. *If either* $\mathrm{Cap}^{(1)}(B^{(1)}) = 0$ *or* $\mathrm{Cap}^{(2)}(B^{(2)}) = 0$, *then* $\mathrm{Cap}(B^{(1)} \times B^{(2)}) = 0$.

(4) *If* f *is* $\mathscr{E}^{(1)}$*-quasi-continuous on* $X^{(1)}$ *and if* g *is* $\mathscr{E}^{(2)}$*-quasi-continuous on* $X^{(2)}$, *then* $f \otimes g$ *is* $\mathscr{E}$*-quasi-continuous on* X.

(5) *Suppose* $\alpha > 0$ *and* $\beta > 0$. *If* $u \in \mathcal{F}^{(1),\mu}$ $(\subset \mathcal{F}^{(1)})$ *is* α*-excessive relative to* $\mathcal{E}^{(1)}$ *and if* $v \in \mathcal{F}^{(2)}$ *is* β*-excessive relative to* $\mathcal{E}^{(2)}$, *then* $u \otimes v$ *is* α*-excessive relative to* $\mathcal{E}^{\beta\nu}$.

Proof. Assertion (1) is immediate from Lemma 3.2. Using the inequality in (4.9), we can show (2), from which (3) and (4) follow. To prove (5), we note that $u \otimes v \in \mathcal{F}^{\nu}$ by (1). For any non-negative element $h \in \mathcal{F}^{\nu} \cap C_0(X)$, we have

$$\begin{aligned}&\mathcal{E}_{\alpha}^{\beta\nu}(u \otimes v, h)\\ &\quad = \int_{X^{(2)}} \mathcal{E}_{\alpha}^{(1)}(u, h(\cdot,\eta))\, v(\eta) dm^{(2)}(\eta) + \int_{X^{(1)}} \mathcal{E}_{\beta}^{(2)}(v, h(\xi,\cdot))\, \tilde{u}(\xi) d\mu(\xi) \geq 0,\end{aligned}$$

using Theorem 4.2 (2) and Lemma 2.2. □

5. Applications

In this section, we turn to the applications of our results to some of the global properties of symmetric Markov processes. We follow the notation used in the previous section.

Theorem 5.1. *If a subset* B *of* X *is an* $\mathcal{E}$*-invariant set, then* B^{η} *are* $\mathcal{E}^{(1)}$*-invariant subsets of* $X^{(1)}$ *for* $m^{(2)}$*-a.a.* η *and* B_{ξ} *are* $\mathcal{E}^{(2)}$*-invariant subsets of* $X^{(2)}$ *for* μ*-a.a.* ξ. *Consequently, if both* $\mathcal{E}^{(1)}$ *and* $\mathcal{E}^{(2)}$ *are irreducible and if* $\mu \neq 0$, *then* $\mathcal{E}$ *is irreducible.*

Proof. We have only to prove the first part. Suppose that B is $\mathcal{E}$-invariant and let $\{\chi_n\}$ be as in Lemma 2.3. Since $\chi_n 1_B \in \mathcal{F}$ for all n, we can assume with no loss of generality that 1_B is $\mathcal{E}$-quasi-continuous. It follows from Lemma 2.3 (1) and Theorem 4.2 that, for any n,

$$\begin{aligned}&\int_{X^{(2)}} \mathcal{E}^{(1)}(\chi_n 1_B(\cdot,\eta), \chi_n 1_{B^c}(\cdot,\eta))\, dm^{(2)}(\eta)\\ &\qquad + \int_{X^{(1)}} \mathcal{E}^{(2)}(\chi_n 1_B(\xi,\cdot), \chi_n 1_{B^c}(\xi,\cdot)) d\mu(\xi) = \mathcal{E}(\chi_n 1_B, \chi_n 1_{B^c}) = 0.\end{aligned}$$

Hence it follows from Lemma 2.3 (2) that

$$\mathcal{E}^{(1)}(\chi_n(\cdot,\eta) 1_{B^{\eta}}, \ \chi_n(\cdot,\eta) 1_{(B^{\eta})^c}) = 0 \qquad (n=1, 2, \ldots) \qquad \text{for } m^{(2)}\text{-a.a. } \eta,$$

$$\mathcal{E}^{(2)}(\chi_n(\xi,\cdot) 1_{B_{\xi}}, \ \chi_n(\xi,\cdot) 1_{B_{\xi}^c}) = 0 \qquad (n=1, 2, \ldots) \qquad \text{for } \mu\text{-a.a. } \xi.$$

This completes the proof of the first part in view of Lemma 2.3 (1). □

Theorem 5.2. (1) *Suppose that* $\mathcal{E}$ *is recurrent. It follows that* $\mathcal{E}^{(1)}$ *is recurrent and, if* $\mu \neq 0$, *that* $\mathcal{E}^{(2)}$ *is also recurrent.*
(2) *If* $\mathcal{E}^{(1)}$ *is transient, then* $\mathcal{E}$ *is transient.*

(3) *If* $\mathscr{E}^{(2)}$ *is transient,* $\mu \neq 0$ *and* $\mathscr{E}^{(1)}$ *is irreducible, then* $\mathscr{E}$ *is transient.*

Proof. Suppose that $\mathscr{E}$ is recurrent. It follows from Lemma 2.4 that $1 \in \mathscr{F}_e$ and $\mathscr{E}(1, 1) = 0$. Hence, it follows from Theorem 4.2 (1) that $1 \in \mathscr{F}_e^{(1)}$, $1 \in \mathscr{F}_e^{(2)}$ and $\int_{X^{(2)}} \mathscr{E}^{(1)}(1, 1) dm^{(2)} + \int_{X^{(1)}} \mathscr{E}^{(2)}(1, 1) d\mu = 0$. Thus, assertion (1) again follows from Lemma 2.4. To prove (2) and (3), suppose $u \in \mathscr{F}_e$ and $\mathscr{E}(u, u) = 0$. It suffices to show that $u = 0$ m-a.e. It follows from Theorem 4.1 that

$$u(\cdot, \eta) \in \mathscr{F}_e^{(1)} \quad \text{and} \quad \| u(\cdot, \eta) \|_{\mathscr{E}^{(1)}} = 0 \quad \text{for } m^{(2)}\text{-a.a. } \eta. \tag{5.1}$$

If $\mathscr{E}^{(1)}$ is transient, then (5.1) implies that $u(\cdot, \eta) = 0$ $m^{(1)}$-a.e. for $m^{(2)}$-a.a. η, and hence, $u = 0$ m-a.e. by Fubini's theorem. This completes the proof of (2). Next, we prove (3). We can assume that $\mathscr{E}^{(1)}$ is recurrent. It follows from Lemma 2.4 (4) that (5.1) implies that, for $m^{(2)}$-almost every η, there exists a constant $c(\eta)$ such that $\tilde{u}(\cdot, \eta) = c(\eta)$ $\mathscr{E}^{(1)}$-q.e., where $\tilde{u}$ denotes an $\mathscr{E}$-quasi-continuous m-version of u. This implies that $\tilde{u} = c$ $\mu \otimes m^{(2)}$-a.e. by Fubini's theorem. Hence it follows from Theorem 4.2 (1) that $c \in \mathscr{F}_e^{(2)}$ and $\| c \|_{\mathscr{E}^{(2)}} = 0$. Since $\mathscr{E}^{(2)}$ is transient, it holds that $c = 0$ $m^{(2)}$-a.e., which implies that $u = c = 0$ m-a.e. □

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