PAPER

Periodic Fourier Transform and Its Application to Wave Scattering from a Finite Periodic Surface

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SUMMARY As a new idea for analyzing the wave scattering and diffraction from a finite periodic surface, this paper proposes the periodic Fourier transform. By the periodic Fourier transform, the scattered wave is transformed into a periodic function which is further expanded into Fourier series. In terms of the inverse transformation, the scattered wave is shown to have an extended Floquet form, which is a 'Fourier series' with 'Fourier coefficients' given by band-limited Fourier integrals of amplitude functions. In case of the TE plane wave incident, an integral equation for the amplitude functions is obtained from the the boundary condition on the finite periodic surface. When the surface corrugation is small, in amplitude, compared with the wavelength, the integral equation is approximately solved by iteration to obtain the scattering cross section. Several properties and examples of the periodic Fourier transform are summarized in Appendix.

key words: *periodic Fourier transform, wave scattering, finite periodic surface*

1. Introduction

This paper deals with a mathematical formulation for the wave scattering from a finite periodic surface. Because any real periodic grating is finite in extent, such a scattering problem is important in practical applications.

When a plane wave is incident on a periodically corrugated surface of infinite extent, the wave is well known to be scattered into discrete directions. Mathematically, the scattered wave is given by the Floquet form, which is a product of a periodic function and an exponential phase factor. Many analytical and numerical works have been carried out on the basis of the Floquet form [1]. However, such a form is valid only for a periodic grating with infinite extent and is no longer applicable to a finite periodic case [2] and semi-infinite periodic case [3], [4], where the surface corrugation has a continuous spectrum. In such cases, the scattered waves are often represented by Fourier integrals [2]–[5]. It seems that a periodic case with infinite extent and a finite periodic case have been considered to be entirely different in mathematical formulation.

The purpose of this paper is to bridge wide gaps between an infinite extended and a finite periodic cases. We introduce a new idea, which is the periodic Fourier transform based on the periodicity of grating. The periodic Fourier transform converts any function $f(x)$ into a spectrum function $F(x, s)$, where $F(x, s)$ is periodic in the x direction and s is a parameter. The inverse transform is given by a Fourier integral with s over a finite interval. Taking the periodic Fourier transform of the scattered wave and expanding the spectrum function into Fourier series, it is shown that the scattered wave has an extended Floquet form, that is a 'Fourier series' with 'Fourier coefficients' given by band-limited Fourier integrals of unknown amplitude functions of s.

As an application of the periodic Fourier transform, we next present a new formulation for the TE wave scattering from a finite corrugated plane shown in Fig. 1. By the periodic Fourier transform, the boundary condition is reduced to an equation involving only periodic functions. Expanding periodic functions into Fourier series, we obtain an integral equation for the amplitude function. When the height of the corrugation is sufficiently small, the integral equation is approximately solved by iteration.

2. Periodic Fourier Transform

As a new idea of analysis, we introduce the periodic Fourier transform, properties and examples of which are summarized in Appendix.

Let us define a displacement operator D associated with a distance L by the relation:

$$Df(x) = f(x + L). \tag{1}$$

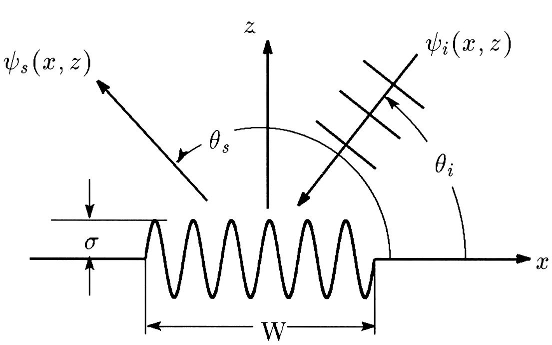

Fig. 1 Scattering and diffraction of a plane wave from a finite periodic surface. The incident plane wave and the scattered wave are denoted by $\psi_i(x, z)$ and $\psi_s(x, z)$, respectively. θ_i is the angle of incidence, θ_s is a scattering angle. W is the width of the corrugation and σ is the height of the corrugation.

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Since $D[Df(x)] = Df(x+L) = f(x+2L) = D^2 f(x)$, D becomes a one parameter group:

$$D^0 = 1(\text{identity}), \quad D^m D^n = D^{n+m}, \\ [D^m]^{-1} = D^{-m}, \tag{2}$$

where m and n are any integers. In terms of the displacement operator D, we define a transformation by the relation,

$$\begin{aligned} F(x,s) &= e^{isx} \sum_{m=-\infty}^{\infty} e^{ismL} D^m f(x) \\ &= e^{isx} \sum_{m=-\infty}^{\infty} e^{ismL} f(x+mL), \end{aligned} \tag{3}$$

which is implicitly assumed to converge. Here, the summation with respect to m is the discrete Fourier transform of the sample sequence $\{f(x+mL),\ m = 0, \pm1, \pm2, \cdots\}$. However, we put the factor e^{isx} to make the spectrum $F(x,s)$ a periodic function of x with the period L. From (3), one easily finds

$$D^m F(x,s) = F(x+mL, s) = F(x,s). \tag{4}$$

Because of this property, we call (3) the periodic Fourier transform and $F(x,s)$ s-periodic function. However, $F(x,s)$ is not periodic with respect to s but it satisfies,

$$F(x, s+k_L) = e^{ik_L x} F(x,s), \quad k_L = \frac{2\pi}{L}, \tag{5}$$

where k_L is the spatial angular frequency of the period L. From (3), we formally find the inverse transform as

$$f(x) = \frac{1}{k_L} \int_{-\pi/L}^{\pi/L} F(x,s) e^{-isx} ds. \tag{6}$$

In (3), x and s may be any numbers. However, (6) and (4) mean that it is sufficient to define $F(x,s)$ over a two dimensional box region with $0 \le x \le L$ and $-k_L/2 \le s \le k_L/2$. We note that the periodic Fourier transform is a simplified and deterministic version of the D^a-Fourier transform developed previously [6]–[9].

3. Application to Scattering Problem

By use of the periodic Fourier transform, this section obtains a form of the scattered wave under the Rayleigh hypothesis.

Let us consider the wave scattering from a finite periodic plane shown in Fig. 1. We write the surface corrugation as

$$z = f(x) = u(x|W) f_p(x) = \begin{cases} 0, & |x| > W/2 \\ f_p(x), & |x| \le W/2 \end{cases}, \tag{7}$$

where $u(x|W)$ is the rectangular pulse defined by (A·13) and $f_p(x)$ is a periodic function with the period L:

$$Df_p(x) = f_p(x) = f_p(x+L). \tag{8}$$

Note that $f(x)$ is implicitly assumed to be a continuous function of x and there are no discontinuities at $x = \pm W/2$. We denote the y component of the electric field by $\psi(x,z)$, which satisfies the wave equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right] \psi(x,z) = 0 \tag{9}$$

in the region $z > f(x)$ and the Dirichlet condition

$$\psi(x,z) = 0, \quad z = f(x) \tag{10}$$

on the surface. We write the incident plane wave $\psi_i(x,z)$ as

$$\psi_i(x,z) = e^{-ipx} e^{-i\beta_0(p)z}, \quad p = k \cdot \cos\theta_i, \tag{11}$$

$$\beta_m(p) = \sqrt{k^2 - (p + mk_L)^2}, \quad Im\,[\beta_m(p)] \ge 0, \\ (m = 0, \pm1, \pm2, \cdots), \tag{12}$$

where θ_i is the angle of incidence. Since the surface is flat for $|x| > W/2$, we put the electric field as

$$\psi(x,z) = e^{-ipx} e^{-i\beta_0(p)z} - e^{-ipx} e^{i\beta_0(p)z} + \psi_s(x,z), \\ \psi_s(x,z) = e^{-ipx} v_s(x,z), \tag{13}$$

where $e^{-ipx} e^{i\beta_0(p)z}$ is the specularly reflected wave and $\psi_s(x,z) = e^{-ipx} v_s(x,z)$ is the scattered wave due to surface deformation. We take the periodic Fourier transform of $v_s(x,z)$:

$$V_s(x,z,s) = e^{isx} \sum_{m=-\infty}^{\infty} v_s(x+mL, z) e^{ismL}, \tag{14}$$

where it should be noted that L is equal to the period of the surface corrugation. Since $V(x,z,s)$ is s-periodic function of x, we write

$$V_s(x,z,s) = \sum_{m=-\infty}^{\infty} A_m(s,z) \exp(-imk_L x). \tag{15}$$

The inverse transform of this gives a form of the scattered wave in the x direction.

$$\psi_s(x,z) = \frac{1}{k_L} \sum_{m=-\infty}^{\infty} \exp(-imk_L x) \\ \times \int_{-\pi/L}^{\pi/L} A_m(s,z) e^{-i(p+s)x} ds. \tag{16}$$

However, we may determine the z-dependence of $A_m(s,z)$ from a fact that $\psi_s(x,z)$ satisfies the wave Eq. (9). Assuming that (16) satisfies (9) term by term, we obtain

$$A_m(s,z) = A_m(s) \exp[i\beta_m(p+s)z] \\ + A_m^{(-)}(s) \exp[-i\beta_m(p+s)z], \tag{17}$$

where $\beta_m(p+s)$ is defined by (12). Here, the first term in the right-hand side physically implies an out-going wave propagating into the z direction and the second one an in-coming wave. When the surface corrugation is small in height and is gentle in slope, however, we simply assume the Rayleigh hypothesis, under which the scattered wave is expressed in terms of out-going waves [1]. Thus, we put

$$V_s(x,z,s) = \sum_{m=-\infty}^{\infty} A_m(s) e^{-imk_L x + i\beta_m(p+s)z}, \quad (18)$$

which satisfies the radiation condition at $z \to \infty$ term by term. Then we obtain a form of the scattered wave under the Rayleigh hypothesis,

$$\psi_s(x,z) = \frac{1}{k_L} \sum_{m=-\infty}^{\infty} e^{-imk_L x} \times \int_{-\pi/L}^{\pi/L} A_m(s) e^{-i(p+s)x + i\beta_m(p+s)z} ds, \quad (19)$$

which is the main result of this paper. Here, $A_m(s)$ is the complex amplitude of the plane wave propagating with wave vector $\mathbf{k} = -(s+p+mk_L)\mathbf{e}_x + \beta_m(p+s)\mathbf{e}_z$, $\mathbf{e}_x$ and $\mathbf{e}_z$ being unit vectors in the x and z directions, respectively. Equation (19) is a 'Fourier series' with 'Fourier coefficients' given by band-limited Fourier integrals of amplitude functions $A_m(s)$.

Roughly speaking, the amplitude $A_m(s)$ has a sharp peak at $s=0$; the height and width of the peak are proportional to W and $2\pi/W$, respectively. As is discussed later, the amplitude $A_m(s)$ becomes proportional to $\delta(s)$ when the width W goes to infinity and the surface becomes a periodic grating with infinite extent,

$$A_m(s) = \hat{A}_m k_L \delta(s), \quad (m = 0, \pm 1, \pm 2, \cdots), \quad (20)$$

where $\hat{A}_m$ is the diffraction amplitude and $\delta(s)$ is Dirac's delta function. In such a limiting case, the form (19) is reduced to the Floquet solution for a periodic grating with infinite extent. Therefore, Eq. (19) should be regarded as an extension of the Floquet solution for a periodic surface with infinite extent.

4. Scattering Cross Section and Optical Theorem

Physically, the diffracted waves are radiated from the corrugated part of the surface and hence they exit only limited regions in space. Restricting our discussions to the far field, however, we only evaluate the integrals in (19) by the saddle point method to get the scattering cross section.

Denoting a scattering angle by θ_s (See Fig. 1), we introduce the polar coordinate:

$$x = r\cos\theta_s, \quad z = r\sin\theta_s. \quad (21)$$

Then, we evaluate (19) by the saddle point method to obtain,

$$\psi_s(r\cos\theta_s, r\sin\theta_s) \approx \frac{k\sin\theta_s}{k_L}\sqrt{\frac{2\pi}{kr}} e^{ikr - i\pi/4} \times \sum_{m=-\infty}^{\infty} A_m(-k\cos\theta_s - p - mk_L) \times u(-k\cos\theta_s - p - mk_L|k_L), \quad (22)$$

where $u(-k\cos\theta_s - p - mk_L|k_L)$ is the rectangular pulse (A$\cdot$13). Form this relation, we obtain the scattering cross section $\sigma(\theta_s|\theta_i)$ per unit length

$$\sigma(\theta_s|\theta_i) = \lim_{r\to\infty} 2\pi \frac{kr}{kW} \cdot |\psi_s(r\cos\theta_s, r\sin\theta_s)|^2 \approx \sum_{m=-\infty}^{\infty} \frac{(2\pi k)^2}{k_L^2 kW} |A_m(-k\cos\theta_s - p - mk_L)|^2 \times \sin^2\theta_s u(-k\cos\theta_s - p - mk_L|k_L), \quad (23)$$

which is a non-dimensional quantity divided by the corrugation width W.

Because the plane wave is incident on the infinitely wide surface, the total incident power is infinite physically. Since the scattering takes place from the corrugated part of the surface, the scattered power always remains finite. We will obtain the optical theorem concerning such a finite quantity of the scattering.

Manipulating the identity $Im[\text{div}\,(\psi^* \text{grad}\,\psi)] = 0$, Im and the asterisk being the imaginary part and the complex conjugate respectively, we obtain the optical theorem,

$$\frac{4\pi}{k_L}\beta_0(p) Re[A_0(0)] = \frac{2\pi}{k_L^2} \sum_{n=-\infty}^{\infty} \int_{-\pi/L}^{\pi/L} Re[\beta_n(p+s)] |A_n(s)|^2 ds \quad (24)$$

$$= \frac{kW}{2\pi} \int_0^{\pi} \sigma(\theta_s|\theta_i) d\theta_s, \quad (25)$$

where Re stands for the real part and we have used (23) to get (25) from (24). Clearly, the right-hand side is the total scattering cross section which is non-negative. Thus, it holds that $Re[A_0(0)] > 0$, where $A_0(0)$ is the complex amplitude of the plane wave scattered into the direction of specular reflection. Since the reflected wave given by the second term in (13) has a negative amplitude, the optical theorem means that the scattering takes place with the loss of specularly reflection component.

5. Integral Equation

In this section, we will obtain an integral equation for the amplitude $A_m(s)$. By the Rayleigh hypothesis, we assume the expansion (19) is valid even on the corrugated part of the surface. Substituting (13) and (19) into the boundary condition (10), we obtain

$$\frac{1}{k_L}\sum_{m=-\infty}^{\infty} e^{-imk_Lx}\int_{-\pi/L}^{\pi/L} A_m(s)e^{-isx+i\beta_m(p+s)f(x)}ds$$
$$= -[e^{-i\beta_0(p)f_p(x)} - e^{i\beta_0(p)f_p(x)}]u(x|W). \quad (26)$$

The exponential factor in this integral is decomposed as

$$e^{i\beta_m(s)f(x)} = 1 + u(x|W)\left[\exp\{i\beta_m(s)f_p(x)\} - 1\right], \quad (27)$$

where the first term 1 in the right-hand side implies the flat surface ($z = f(x) \equiv 0$) and the second term is the effect of variation from the flat surface. By use of this decomposition, we obtain

$$\frac{1}{k_L}\sum_{m=-\infty}^{\infty} e^{-imk_Lx}\int_{-\pi/L}^{\pi/L} A_m(s)e^{-isx}ds$$
$$+\frac{1}{k_L}\sum_{m=-\infty}^{\infty} e^{-imk_Lx}\int_{-\pi/L}^{\pi/L} A_m(s)$$
$$\times e^{-isx}\left[e^{i\beta_m(p+s)f_p(x)} - 1\right]u(x|W)ds$$
$$= 2i\sin[\beta_0(p)f_p(x)]u(x|W). \quad (28)$$

As is seen in (A·4), the periodic Fourier transform of a product of a periodic function and a weighting function becomes a product of the periodic function and the periodic Fourier transform of the weighting function. In other words, periodic factors such as $f_p(x)$, e^{-imk_Lx}, $\exp[i\beta_m(p+s)f_p(x)]$ and $\sin[\beta_0(p)f_p(x)]$ are all invariant under the periodic Fourier transform. Taking this property and using (A·15) and (A·2), we calculate the periodic Fourier transform of (28) to obtain,

$$\sum_{m=-\infty}^{\infty} e^{-imk_Lx}A_m(s) + \frac{1}{k_L}\sum_{m=-\infty}^{\infty} e^{-imk_Lx}$$
$$\times\int_{-\pi/L}^{\pi/L} A_m(s')\left[e^{i\beta_m(p+s')f_p(x)} - 1\right]F_u(x, s-s'|W)ds'$$
$$= 2i\sin[\beta_0(p)f_p(x)]F_u(x, s|W), \quad (29)$$

where $F_u(x, s|W)$ describes the effect of the corrugation width W. Our formulation using the periodic Fourier transform has an advantage such that the boundary condition (10) is reduced to an equation involving only periodic functions of x. In fact, $f_p(x)$, e^{-imk_Lx} and $F_u(x, s|W)$ are all periodic functions of x with the period L. These periodic functions are expanded into Fourier series,

$$\exp\{i\sigma\beta_m(p)f_p(x)\} = \sum_{l=-\infty}^{\infty} K_l(p + mk_L)e^{ilk_Lx},$$

$$\sin[\sigma\beta_0(p)f_p(x)] = \frac{1}{2i}\sum_{l=-\infty}^{\infty}[K_l(p) - K^*_{-l}(p)]e^{ilk_Lx},$$

$$K_l(p) = \frac{1}{L}\int_{-L/2}^{L/2} e^{i\sigma\beta_0(p)f_p(x)}e^{-ilk_Lx}dx, \quad (30)$$

where the angle θ_i of incidence and $\beta_0(p)$ are assumed to be real. Then we finally obtain an integral equation for the amplitude $A_m(s)$ as

$$A_m(s) + \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi/L}^{\pi/L}$$
$$\cdot C(s + mk_L|s' + nk_L)A_n(s')ds'$$
$$= \frac{1}{L}\sum_{l=-\infty}^{\infty}[K_l(p) - K^*_{-l}(p)]$$
$$\cdot U\left(s + (m+l)k_L|W\right), \quad (31)$$

$$C(s + mk_L|s' + nk_L)$$
$$= \sum_{l=-\infty}^{\infty} U\left(s - s' + (m-n+l)k_L|W\right)$$
$$\times [K_l(p + s' + nk_L) - \delta(l, 0)], \quad (32)$$

where $U(s|W)$ is the Fourier transform of $u(x|W)$ and is given by (A·16).

Let us consider a limiting case where W goes to infinity and the surface becomes a periodic grating with infinite extent. From (A·15), (A·16) and (A·17), we obtain

$$\lim_{W\to\infty} F_u(x, s|W) = \frac{2\pi}{L}\delta(s) = k_L\delta(s), \quad |s| \le \frac{k_L}{2}. \quad (33)$$

By this relation and (29), one easily finds that $A_m(s)$ is proportional to $\delta(s)$. If we define the diffraction amplitude $\hat{A}_m$ by (20), the condition (29) is reduced to

$$\sum_{m=-\infty}^{\infty}\hat{A}_m e^{-imk_Lx+i\beta_m(p)f_p(x)} = 2i\sin[\beta_0(p)f_p(x)], \quad (34)$$

which is a conventional equation determining the diffraction amplitude $\hat{A}_m$. From this example, it is concluded again that the form (19) is considered as an extension of the Floquet solution.

6. Sinusoidal Case

Let us consider a case where the surface corrugation is sinusoidal,

$$f_p(x) = \sigma\cdot\sin(k_Lx). \quad (35)$$

Here, σ is a small height parameter with $k\sigma \ll 2\pi$. Since $f(x)$ is continuous at $x = \pm W/2$, the width W should be $W = nL$, n being any positive integer. Then, one easily finds

$$K_l(p) = \frac{1}{L}\int_{-L/2}^{L/2}\exp\{i\sigma\beta_0(p)\sin(k_Lx)\}$$

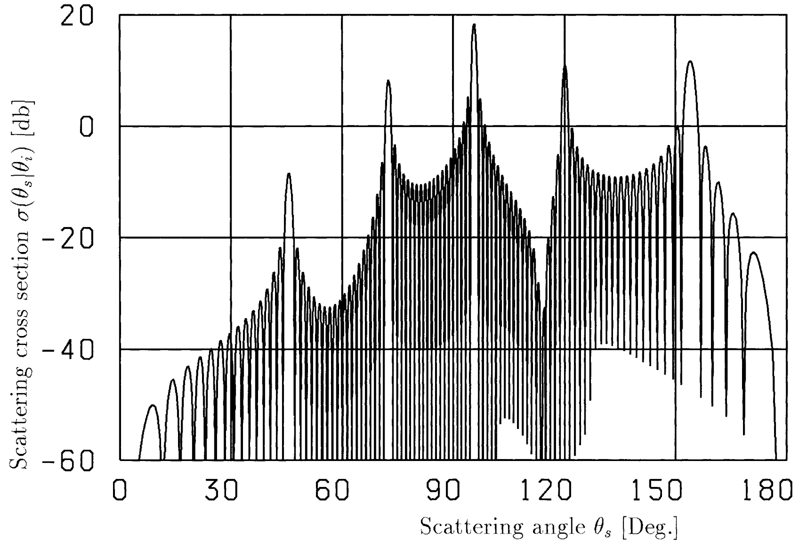

Fig. 2 Scattering cross section $\sigma(\theta_s|\theta_i)$ for a finite sinusoidal surface with $M_t = 6$, $L = 2.5\lambda$, $W = 20L = 50\lambda$, $\sigma = 0.1\lambda$ and $\theta_i = \pi/3$, λ being the wavelength. Peaks at scattering angles 45.6°, 72.5°, 95.7°, 120.00° and 154.2° are effects of diffraction by periodic corrugation. The side-lobes are interferences of waves diffracted from the ends of corrugation at $x = \pm W/2$.

$$\times \exp(-ilk_L x)\, dx = J_l(\sigma\beta_0(p)), \qquad (36)$$

where $J_l(\cdot)$ stands for Bessel function. The coefficient $K_l(p)$ is a small quantity of the order of $O(\sigma^l)$. From this fact and (32), we find that $C(s + mk_L|s' + nk_L)$ is the order of $O(\sigma^1)$.

It is still open question to find out an efficient method for solving (31). For a sufficiently small σ, however, the integral Eq. (31) may be approximately solved by iteration using an initial guess:

$$A_m(s) = \frac{2}{L} \sum_{l=-\infty}^{\infty} J_{2l+1}(\sigma\beta_0(p)) \times U(s + (m + 2l + 1)k_L|W), \qquad (37)$$

and putting the higher order amplitudes equal to zero:

$$A_m(s) \equiv 0, \quad |m| \geq M_t \qquad (38)$$

where M_t is the order of truncation. Here, (37) is derived from the right-hand side of (31). In order to estimate the accuracy of the iterative solution, we define the error E_{rr} with respect to the optical theorem as

$$E_{rr} = 1 - \frac{P_c}{P_s}, \quad P_c = \frac{4\pi}{k_L}\beta_0(p)Re[A_0(0)],$$

$$P_s = \frac{kW}{2\pi}\int_0^{\pi} \sigma(\theta_s|\theta_i)d\theta_s. \qquad (39)$$

Putting $M_t = 6$, $L = 2.5\lambda$, $W = 20L = 50\lambda$, $\sigma = 0.1\lambda$ and $\theta_i = \pi/3$, λ being the wavelength, we solved (31) by iteration. Then, it was found that the error $|E_{rr}|$ decreases when the number of iteration increases. The error reaches to $|E_{rr}| = 1.73 \times 10^{-4}$ by 10 times of iteration, at which $P_c = 123.8972$ and $P_s = 123.9187$. Using such iterative solution, we calculate the scattering cross section $\sigma(\theta_s|\theta_i)$ as a function of θ_s shown in Fig. 2. In Fig. 2, there are five major peaks at scattering angles 45.6°, 72.5°, 95.7°, 120.0° and 154.2°. These angles agree with the diffraction angles calculated from the famous grating formula: $\theta_s = \cos^{-1}(-\cos(\theta_i) - m/L)$, $(m = 0, \pm1, \pm2, \cdots)$, m being the order of diffraction. Thus, we may conclude that major peaks are effects of diffraction due to periodic corrugation. However, we see a lot of side-lobes around the major peaks, which are mathematically caused by the rectangular function $u(x|W)$ in the the surface deformation (7). Physically, the side-lobes are considered as interferences of edge diffracted waves radiated from the ends of the corrugation at $x = \pm W/2$.

7. Conclusions

As a new idea for analyzing the wave scattering and diffraction from a finite periodic surface, we have proposed the periodic Fourier transform. Then, the scattered wave is shown to have an extended Floquet form, which is a 'Fourier series' with 'Fourier coefficients' given by band-limited Fourier integrals of amplitude functions. By the periodic Fourier transform, the boundary condition on the finite periodic surface is transformed into an equation involving only periodic functions. Expanding these periodic functions into Fourier series, the boundary condition is finally reduced to an integral equation for the amplitude functions. Assuming that the surface corrugation is small compared with the wavelength, we approximately solved the integral equation by iteration.

However, the periodic Fourier transform is defined formally but no rigorous mathematical discussions are given. Also, our discussion was limited to a TE wave case with Rayleigh hypothesis. However, we note that our formulation can be immediately applied to TM wave case and the wave scattering from a dielectric wave guide with a finite periodic corrugation [10]. However, it is pointed out that any real grating is finite in extent and random in various degrees. If a real grating is modeled by the periodic stationary process in the probability theory [11], [12], such a finite and random case may be formulated by a combination of the periodic Fourier transform and the D^a Fourier transform [6], [7]. It is much interesting to extend the periodic Fourier transform into two and three dimensional cases. However, these problems are left for future study.

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Appendix: Periodic Fourier Transformation

This appendix summarizes some properties of the periodic Fourier transform. For simplicity, we will denote the periodic Fourier transform and its inverse by the symbol: $f(x) \Longleftrightarrow F(x,s)$.

linear transform. If $f(x) \Longleftrightarrow F(x,s)$ and $g(x) \Longleftrightarrow G(x,s)$, then

$$\alpha f(x) + \beta g(x) \Longleftrightarrow \alpha F(x,s) + \beta G(x,s), \tag{A·1}$$

where α and β are any constants.

modulation and shift. If $f(x) \Longleftrightarrow F(x,s)$, then

$$\begin{aligned} &f(x)e^{iqx} \Longleftrightarrow F(x, s+q), \\ &f(x-x_0) \Longleftrightarrow e^{isx_0} F(x-x_0, s). \end{aligned} \tag{A·2}$$

product of weighting function and periodic function Let $w(x)$ and $f_p(x)$ be a weighting function and a periodic function with $f_p(x) = f_p(x+L)$, respectively. If we write

$$w(x) \Longleftrightarrow F_w(x,s), \tag{A·3}$$

then, a product $f_p(x)w(x)$ is transformed into a product of the periodic function and the periodic Fourier transform of the weighting function

$$f_p(x)w(x) \Longleftrightarrow f_p(x)F_w(x,s), \tag{A·4}$$

which means that a periodic factor is invariant under the periodic Fourier transform. This is an important property of the periodic Fourier transform.

relation with Fourier spectrum Let $\hat{F}(s)$ be the Fourier spectrum of $f(x)$. Then, we find

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\hat{F}(s)ds \\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-imk_L x}\int_{-k_L/2}^{k_L/2} e^{-isx}\hat{F}(s+mk_L)ds. \end{aligned} \tag{A·5}$$

Comparing (A·5) with (6), we formally obtain the relation of $F(x,s)$ with the Fourier spectrum $\hat{F}(s)$ as

$$\frac{1}{k_L}F(x,s) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-imk_L x}\hat{F}(s+mk_L). \tag{A·6}$$

inner product and Perseval's theorem We put

$$\begin{aligned} f(x) &\Longleftrightarrow F(x,s) = \sum_{m=-\infty}^{\infty} e^{-imk_L x}F_m(s), \\ g(x) &\Longleftrightarrow G(x,s) = \sum_{m=-\infty}^{\infty} e^{-imk_L x}G_m(s). \end{aligned} \tag{A·7}$$

Then, we find the inner product of $f(x)$ and $g(x)$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)g^*(x)dx$$

$$= L \int_{-k_L/2}^{k_L/2} ds \int_{-L/2}^{L/2} F(x,s) G^*(x,s) dx$$
$$= L^2 \sum_{m=-\infty}^{\infty} \int_{-k_L/2}^{k_L/2} F_m(s) G_m^*(s) ds, \qquad \text{(A· 8)}$$

where the asterisk denotes the complex conjugate. If we put $f(x) = g(x)$ in (A· 8), we obtain Perseval's theorem,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx = L \int_{-k_L/2}^{k_L/2} ds \int_{-L/2}^{L/2} |F(x,s)|^2 dx$$
$$= L^2 \sum_{m=-\infty}^{\infty} \int_{-k_L/2}^{k_L/2} |F_m(s)|^2 ds. \qquad \text{(A· 9)}$$

example 1, constant. $f_c(x) = 1$.

$$1 \Longleftrightarrow F_c(x,s), \qquad \text{(A· 10)}$$

$$F_c(x,s) = e^{isx} \sum_{m=-\infty}^{\infty} e^{ismL}$$
$$= k_L e^{isx} \sum_{m=-\infty}^{\infty} \delta(s + mk_L)$$
$$= k_L \sum_{m=-\infty}^{\infty} e^{-imk_L x} \delta(s + mk_L), \qquad \text{(A· 11)}$$

where $k_L = 2\pi/L$ and $\delta(\cdot)$ stands for the Dirac δ function. Here, we have used the identity

$$\sum_{m=-\infty}^{\infty} e^{iLms} = 2\pi \sum_{m=-\infty}^{\infty} \delta(Ls + 2\pi m)$$
$$= \frac{2\pi}{L} \sum_{m=-\infty}^{\infty} \delta\left(s + \frac{2\pi m}{L}\right). \qquad \text{(A· 12)}$$

example 2. rectangular pulse If $u(x|W)$ is a rectangular pulse or gate function

$$u(x|W) = u^2(x|W) = \begin{cases} 1, & |x| \le W/2 \\ 0, & |x| > W/2 \end{cases}, \qquad \text{(A· 13)}$$

$$u(x|W) \Longleftrightarrow F_u(x,s|W), \qquad \text{(A· 14)}$$

$$F_u(x,s|W) = \frac{1}{L} \sum_{m=-\infty}^{\infty} U(s + mk_L|W) e^{-imk_L x}, \qquad \text{(A· 15)}$$

where W is the pulse width and $U(s|W)$ is the Fourier transform of $u(x|W)$,

$$U(s|W) = \int_{-W/2}^{W/2} e^{isx} u(x|W) dx$$
$$= W \frac{\sin\left(\frac{Ws}{2}\right)}{\left(\frac{Ws}{2}\right)}, \qquad \text{(A· 16)}$$

$$\lim_{W \to \infty} U(s|W) = 2\pi\delta(s). \qquad \text{(A· 17)}$$

example 3. periodic function with the period L. If $f_p(x)$ is a periodic function satisfying $f_p(x) = f_p(x + L)$, then it holds that

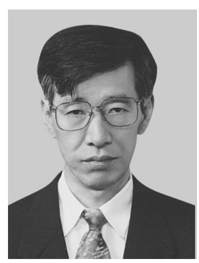
$$f_p(x) = f_p(x) \cdot 1 \Longleftrightarrow f_p(x) F_c(x,s), \qquad \text{(A· 18)}$$

where 1 in the left-hand side should be understood as a weighting function and $F_c(x,s)$ is the periodic Fourier transform of 1.

example 4. product of rectangular pulse and a periodic function. If $u(x|W)$ is the rectangular pulse (A· 13) and $f_p(x)$ is a periodic function with $f_p(x) = f_p(x + L)$, then,

$$f_p(x) u(x|W) \Longleftrightarrow f_p(x) F_u(x,s|W) \qquad \text{(A· 19)}$$

where $F_u(x,s|W)$ is defined by (A· 15).



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