【学術論文】（査読あり）
京都工芸繊維大学学術報告書　第 7 巻（平成 26 年 8 月）
BULLETIN OF KYOTO INSTITUTE OF TECHNOLOGY　7（Aug. 2014）

A characterization of the Brownian ratchet by a Skorohod-type equation

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（2013 年 12 月 25 日原稿受理、2014 年 6 月 17 日採用決定）

SUMMARY

We formulate using a Skorohod-type equation a process with the following properties and call it a Brownian ratchet with the integer-valued moving boundary(BRIMB). It behaves like a Brownian motion reflected upward at a boundary, called the moving boundary, before it arrives at the point with the distance 1 from the moving boundary. When it arrives, the moving boundary jumps upward by 1. Afterward these processes continue this algorithm, which justifies the word ratchet. We study existence of the process, uniqueness of the law, and some distributional properties.

MSC(2000) Primary 60J65; Secondary 60K05

Keywords: Brownian ratchet, Brownian motion, Skorohod equation, Renewal theory.

Introduction and the Main Theorem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space and let β_t be an $(\mathcal{F}_t)$-Brownian motion starting from 0 in the sense of Definition III(2.20) of Revuz and Yor(1999)[8] or equivalently, let β_t be an Brownian motion starting from 0 and let $(\mathcal{F}_t)$ be the filtration for β_t in the sense of Definition 2.1.1 of Karatzas and Shreve(1991)[6]. In this note, we consider the following conditions (C.1)–(C.5) concerning an $(\mathcal{F}_t)$-adapted $\mathbb{R}^3$-valued process (B_t, R_t, A_t) such that B_t is continuous; R_t is $\mathbb{Z}_+$-valued, nondecreasing, and right-continuous; and A_t is nondecreasing and continuous.

(C.1): $B_0(\omega) = R_0(\omega) = A_0(\omega) = 0$ a.s.
(C.2): $B_t(\omega) \geq R_t(\omega)$ for all $t \in [0, \infty)$ a.s.
(C.3): $R_t(\omega) - R_{t-}(\omega) = 1_{\{B_t(\omega)=1+R_{t-}(\omega)\}}$ for all $t \in (0, \infty)$ a.s.
(C.4): $B_t(\omega) = \beta_t(\omega) + A_t(\omega)$ for all $t \in [0, \infty)$ a.s.
(C.5): $\displaystyle\int_0^\infty 1_{\{B_s(\omega)-R_s(\omega)>0\}} dA_s(\omega) = 0$ a.s.

We call (B_t, R_t, A_t) *a Brownian ratchet with the integer-valued moving boundary*(BRIMB) driven by β_t.

Let us explain the behavior of this process intuitively. When B_t moves in the interval $[R_t, 1 + R_t)$, it behaves like a Brownian motion reflecting at the boundary R_t. Each time B_t hits $1 + R_{t-}$, the boundary jumps by 1 so that B_t will be reflected at this new boundary; such behavior can be described with the phrase "ratchet and moving boundary."

Such a process is brought to our attention by Kiyotaka Suzaki(a private communication) who explained a problem concerning harmonic measures for leafwise diffusion processes on foliated spaces and gave an intuitive description of the process as in the last paragraph. He is interested in the ergodic theory of foliated spaces, while we are interested in the possibility of characterizing the process by a system of stochastic equations. In this note, we formulate the BRIMB as a triplet (B_t, R_t, A_t), where the third process A_t is the sojourn time density at R_t for B_t. The conditions (C.4) and (C.5) describe the reflection at R_t in the same way as the well-known Skorohod equation does. The condition (C.3) has an interesting feature that the stochastic differential of R_t is not given by the differential of any other process but by some indicator function.

The dynamics of BRIMB can also be regarded as the extreme case of Brownian ratchets, which are charmingly illustrated in §46.1 of Feynman et al.(1963)[4)] and have motivated a body of literature in both physics and biology. While a spectrum of different dynamics of Brownian ratchets have been studied, we refer the reader to Budhiraja and Fricks(2006)[2)] who construct a process by concatenation of independent diffusion processes with a.s. finite life times. In ref. 2), they also prove that a sequence of discrete processes converges weakly to the process obtained by concatenation but they are not based on the characterizing equation for the whole process.

In Theorem 1, we state existence of BRIMB and uniqueness in law for BRIMB characterized by (C.1)–(C.5). We prove Theorem 1 in §2 and we apply the renewal theory to reveal some elementary properties of BRIMB in §3.

Theorem 1. (i) *On any probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ endowed with an $(\mathcal{F}_t)$-Brownian motion β_t, there exists a BRIMB (B_t, R_t, A_t) driven by β_t.*

(ii) *A BRIMB (B_t, R_t, A_t) driven by β_t is adapted to the natural filtration $(\mathcal{F}_t^\beta)$ of β_t. The law of BRIMB is unique.*

(iii) *Let (B_t, R_t, A_t) be a BRIMB driven by β_t and let $\sigma_n = \inf\{t \geq 0 | R_t = n\}$ for a nonnegative integer n. Then σ_n is equal to* $\inf\{t \geq 0 | B_t = n\}$ *and the finite-horizon processes*

$(B_{\sigma_n+t} - n; t \in [0, \sigma_{n+1} - \sigma_n])$ *are independent copies of the absolute value of a Brownian motion that is started from* 0 *and killed upon hitting* ± 1.

Remark 1. In Theorem 1(i), we construct BRIMB in a similar way to Definition 1 of Budhiraja and Fricks[2)] who construct a diffusion ratchet.

Theorem 1(iii) is a converse to Definition 1 of ref. 2) for the case $(a(x), b(x)) \equiv (1, 0)$ and hence the Brownian ratchet defined in this case in ref. 2) has the same law as B_t of BRIMB.

Proof of the theorem

Proof of Theorem 1(i).

In the first half of the proof, we construct (B_t, R_t, A_t) and a sequence $(\sigma_n)_{n\in\mathbb{Z}_+}$ of stopping times on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where $\mathbb{Z}_+$ is the set of all nonnegative integers.

We begin by setting $m_t^{(0)} = \max\{-\beta_s | s \in [0, t]\}$ for $t \in [0, \infty)$. The process $\beta_t + m_t^{(0)}$ is a reflected Brownian motion started from 0 and is reflected upward at 0 such that $\int_0^\infty 1_{\left\{\beta_s + m_s^{(0)} > 0\right\}} dm_s^{(0)} = 0$ a.s. We set $\sigma_1 = \inf\left\{t \geq 0 \,\middle|\, \beta_t + m_t^{(0)} = 1\right\}$. Note that $\beta_t + m_t^{(0)}$ is adapted to $(\mathcal{F}_t^\beta)$ and σ_1 is a $(\mathcal{F}_t^\beta)$-stopping time.

We define $\sigma_0 = 0$ for the sake of consistency of notation and, for $t \in [0, \sigma_1]$, set $B_t = \beta_t + m_t^{(0)}$ and $A_t = m_t^{(0)}$. We set $R_t = 0$ for $t \in [0, \sigma_1)$ and $R_{\sigma_1} = 1$. It follows that $B_{\sigma_1} = 1$ from this definition.

For any $n \in \mathbb{N}$, we assume that a stopping time σ_n and a process $((B_t, R_t, A_t); t \in [0, \sigma_n])$ are constructed and it holds $R_{\sigma_n} = B_{\sigma_n} = n$. Note that the process $(\beta_t - \beta_{\sigma_n}; t \in [\sigma_n, \infty))$ is a Brownian motion started from 0 at the instant σ_n, where we consider that any process starting at a random instant stays at an extra point, outside $\mathbb{R}$, before that instant. We hold this convention throughout this note.

For $t \in [\sigma_n, \infty)$, we set

$$m_t^{(n)} = \max\{-(\beta_s - \beta_{\sigma_n}) | s \in [\sigma_n, t]\}. \tag{1}$$

The process $(\beta_t - \beta_{\sigma_n} + m_t^{(n)}; t \in [\sigma_n, \infty))$ is a reflected Brownian motion started from 0 at the instant σ_n that is reflected upward at 0. Moreover, it is adapted to $(\mathcal{F}_t^\beta)$ and satisfies

$$\int_{\sigma_n}^\infty 1_{\left\{\beta_s - \beta_{\sigma_n} + m_s^{(n)} > 0\right\}} dm_s^{(n)} = 0 \tag{2}$$

a.s. We set

$$\sigma_{n+1} = \inf\left\{t \geq \sigma_n \,\middle|\, \beta_t - \beta_{\sigma_n} + m_t^{(n)} = 1\right\}, \tag{3}$$

which is an $(\mathcal{F}_t^\beta)$-stopping time.

We define, for $t \in (\sigma_n, \sigma_{n+1}]$,

$$B_t = n + \beta_t - \beta_{\sigma_n} + m_t^{(n)} \text{ and } A_t = A_{\sigma_n} + m_t^{(n)}. \tag{4}$$

Note that these equations hold for $t = \sigma_n$ by definition, which implies continuity over the time interval $[\sigma_n, \sigma_{n+1}]$. It also holds that A_t is nondecreasing and $B_{\sigma_{n+1}} = n + 1$. We set

$$R_t = n \text{ for } t \in (\sigma_n, \sigma_{n+1}) \text{ and } R_{\sigma_{n+1}} = n + 1. \tag{5}$$

Since $R_{\sigma_n} = n$, R_t is right-continuous. As well, it is $\mathbb{Z}_+$-valued and nondecreasing.

The processes constructed above are adapted to $(\mathcal{F}_t^\beta)$. Moreover, it is straightforward to see that (1)–(5) are originally presented for $n \in \mathbb{N}$ but are valid for all $n \in \mathbb{Z}_+$.

In the second half of the proof, we prove that (B_t, R_t, A_t) constructed above satisfy (C.1)–(C.5). The condition (C.1) is clear from the definition.

To verify (C.2), assume $n \in \mathbb{Z}_+$ and $t \in [\sigma_n, \sigma_{n+1})$. By (5) and (1), it holds $R_t = n$ and $m_t^{(n)} \geq -(\beta_t - \beta_{\sigma_n})$. Hence $B_t = n + \beta_t - \beta_{\sigma_n} + m_t^{(n)} \geq n = R_t$. Since $\cup_{n \in \mathbb{Z}_+} [\sigma_n, \sigma_{n+1}) = \mathbb{R}$, we have (C.2).

The condition (C.3) is verified if we prove both sides are equal to 0 for any $n \in \mathbb{Z}_+$ and any $t \in (\sigma_n, \sigma_{n+1})$, and that they are equal to 1 for $t = \sigma_n$ and any $n \in \mathbb{N}$.

If $t \in (\sigma_n, \sigma_{n+1})$, it holds on one hand that $R_t = R_{t-} = n$ and hence the left hand side of (C.3) is 0. On the other hand, it follows from (3) that $0 \leq \beta_t - \beta_{\sigma_n} + m_t^{(n)} < 1$ and hence (4) implies $B_t < n + 1$ and the right hand side of (C.3) is 0.

If $t = \sigma_n$ for some $n \in \mathbb{N}$, we have $B_t = R_t = n$ and $R_{t-} = n - 1$ by (5). Hence both sides of (C.3) are equal to 1.

To verify (C.4), assume that it holds for all $t \in [0, \sigma_n]$. Since $n = B_{\sigma_n} = \beta_{\sigma_n} + A_{\sigma_n}$, we have, for any $t \in (\sigma_n, \sigma_{n+1}]$, $B_t = n + \beta_t - \beta_{\sigma_n} + m_t^{(n)} = \beta_t + A_{\sigma_n} + m_t^{(n)} = \beta_t + A_t$ by (4).

The condition (C.5) holds since it follows from (2) that

$$0 \leq \int_{\sigma_n}^{\sigma_{n+1}} 1_{\{B_s(\omega) - R_s(\omega) > 0\}} dA_s(\omega) \leq \int_{\sigma_n}^{\infty} 1_{\left\{\beta_s - \beta_{\sigma_n} + m_s^{(n)} > 0\right\}} dm_s^{(n)} = 0$$

for any $n \in \mathbb{Z}_+$. □

Proof of Theorem 1(ii).

Let (B_t, R_t, A_t) be a BRIMB driven by β_t. We set $\sigma_0 = 0$ and, for any $n \in \mathbb{N}$, set

$$\sigma_n = \inf \{t \geq 0 | R_t = n\}. \tag{6}$$

We begin by proving inductively that

$$\sigma_n = \inf \{t \geq 0 | B_t = n\}. \tag{7}$$

for all $n \in \mathbb{Z}_+$, which trivially holds for $n = 0$.

Assume that (7) holds for some $n \in \mathbb{Z}_+$.

If $\sigma_n < t < \sigma_{n+1}$, it holds that $R_t = R_{t-} = n$ by the definition (6) and hence the left hand side of (C.3) is 0. Observing the right hand side of (C.3), we conclude that $B_t \neq n + 1$, in fact, $B_t < n + 1$.

If $t = \sigma_{n+1}$, it holds $R_t = n+1$ and $R_{t-} = n$. Since the both sides of (C.3) is 1, we have $B_t = n+1$.

These two observations imply $\sigma_{n+1} = \inf\{t \geq 0 | B_t = n+1\}$ and the proof of (7) is complete.

We next make use of the Skorohod equation over random periods. We adapt Lemma 3.6.14 in Karatzas and Shreve(1991)[6] for any time interval and present it as Lemma 1. Note that these lemmas are meant for deterministic functions.

Lemma 1. *Let $-\infty < a < b < \infty$ and $y(\cdot) = \{y(t); a \leq t \leq b\}$ be a continuous function with $y(a) = 0$.*

Then there exists a unique continuous function $k(\cdot) = \{k(t); a \leq t \leq b\}$ such that

(i) $x(t) := y(t) + k(t) \geq 0$ *for any* $t \in [a, b]$,

(ii) $k(a) = 0$, $k(\cdot)$ *is nondecreasing, and*

(iii) $k(\cdot)$ *is flat off* $\{t \in [a, b] | x(t) = 0\}$*; i.e.,* $\int_a^b 1_{\{x(s)>0\}} dk(s) = 0$.

Moreover, this function is given by $k(t) = \max\{-y(s) | s \in [a, t]\}$ *for* $t \in [a, b]$.

One can easily modify the proof in p.210 of ref. 6) for the time interval $[a, b]$, so we omit the proof. We set $a = \sigma_n$, $b = \sigma_{n+1}$, $y(t) = \beta_t - \beta_{\sigma_n}$, $k(t) = A_t - A_{\sigma_n}$, and $x(t) = B_t - n$ in this Lemma and verify the conditions (i)–(iii).

Note that $\beta_{\sigma_n} + A_{\sigma_n} = B_{\sigma_n} = n$. Let $\sigma_n \leq t \leq \sigma_{n+1}$. We have $B_t - n \geq R_t - n \geq 0$ by (C.2) and $B_t - n = (\beta_t - \beta_{\sigma_n}) + (A_t - A_{\sigma_n})$ by (C.4). Since A_t is continuous, it follows from (C.5) that $\int_{\sigma_n}^{\sigma_{n+1}} 1_{\{B_s > n\}} d(A_s - A_{\sigma_n}) = 0$. The conditions (i)–(iii) are thus verified.

Now the conclusion of Lemma 1 is

$$A_t - A_{\sigma_n} = \max\{-(\beta_s - \beta_{\sigma_n}) | s \in [\sigma_n, t]\} \tag{8}$$

for all $t \in [\sigma_n, \sigma_{n+1}]$. Hence we can conclude inductively that, for any $n \in \mathbb{Z}_+$, σ_{n+1} is a stopping time w.r.t. $\mathcal{F}_t^\beta$ and that $A_{t\wedge\sigma_{n+1}}$ and $B_{t\wedge\sigma_{n+1}} = \beta_{t\wedge\sigma_{n+1}} + A_{t\wedge\sigma_{n+1}}$ are adapted to $\mathcal{F}_{t\wedge\sigma_{n+1}}^\beta$. We also have

$$\sigma_{n+1} - \sigma_n = \inf\{t \geq 0 | (\beta_{\sigma_n+t} - \beta_{\sigma_n}) + (A_{\sigma_n+t} - A_{\sigma_n}) = 1\}$$

by (7). This implies that $(\sigma_{n+1} - \sigma_n)_{n\in\mathbb{Z}_+}$ is a sequence of i.i.d. strictly positive random variables and hence $\lim_{n\to\infty} \sigma_n = \infty$.

Taking the limit of $\left(B_{t\wedge\sigma_{n+1}}, A_{t\wedge\sigma_{n+1}}\right)$ as $n \to \infty$, we deduce that (B_t, A_t) is adapted to $\mathcal{F}_t^\beta$. It follows from $R_t = \max\{n \geq 0 | \sigma_n \leq t\}$ that R_t is adapted to $\mathcal{F}_t^\beta$. Being a functional of β_t, the law of (B_t, R_t, A_t) is unique. □

Proof of Theorem 1(iii).

Let $\tilde{\beta}_t^{(n)} = \beta_{\sigma_n+t} - \beta_{\sigma_n}$ and $\tilde{B}_t^{(n)} := B_{\sigma_n+t} - n$ for any $n \in \mathbb{Z}_+$ and $t \geq 0$. Then the process $\left(\tilde{\beta}_t^{(n)}; t \geq 0\right)$ is a Brownian motion started from 0 and is independent of $\mathcal{F}_{\sigma_n}^\beta$.

By (8) and (C.4), we have $\tilde{B}_t^{(n)} = \tilde{\beta}_t^{(n)} + \max\left\{-\tilde{\beta}_s^{(n)} | s \in [0, t]\right\}$ and hence $\left(\tilde{B}_t^{(n)}; t \geq 0\right)$ is a reflected Brownian motion independent of $\mathcal{F}_{\sigma_n}^\beta$. Now (7) is equivalent to $\sigma_{n+1} -$

$\sigma_n = \inf\left\{t \geq 0 | \tilde{B}_t^{(n)} = 1\right\}$, which implies that $\left(\tilde{B}_t^{(n)}; t \in [0, \sigma_{n+1} - \sigma_n]\right)$ is measurable w.r.t. $\mathcal{F}^\beta_{\sigma_{n+1}}$, independent of $\mathcal{F}^\beta_{\sigma_n}$, and is a reflected Brownian motion started from 0 and killed upon hitting 1. The independence between them follows immediately. □

Application of the renewal theory

Since the path of B_t is a concatenation of i.i.d. paths, it may be natural to remark some consequence of the well-known renewal theory that can be found in Feller(1966)[3)]. The renewal theory is also applied in Theorem 2 of ref. 2) to study the average speed of the Brownian ratchet in the periodic case.

We first collect known facts concerning the common distribution of $(\tilde{B}_t^{(n)}; t \in [0, \sigma_{n+1} - \sigma_n])$ defined in the proof of Theorem 1(iii). Let

$$G(t, s; x) = P\left[\, \tilde{B}_t^{(n)} \leq x, \sigma_{n+1} - \sigma_n > t + s \,\right]$$

for $0 \leq x \leq 1$, $t \geq 0$, and $s \geq 0$ and let $f(t)$ be the density for the distribution of $\sigma_{n+1} - \sigma_n$. We can deduce from a formula in p.105 of Borodin and Salminen(1996)[1)] that

$$\begin{aligned} G(t,0;x) &= \int_{-x}^{x} dy \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \left(\exp\left(-\frac{(y+4n)^2}{2t}\right) - \exp\left(-\frac{(y+4n+2)^2}{2t}\right)\right) \\ &= \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \exp\left(-\frac{(2m+1)^2\pi^2}{8} t\right) \sin\left(\frac{2m+1}{2}\pi x\right), \\ f(t) &= \sqrt{\frac{2}{\pi t^3}} \sum_{n=-\infty}^{\infty} (4n+1) \exp\left(-\frac{(4n+1)^2}{2t}\right) \\ &= \sum_{m=0}^{\infty} \frac{\pi(2m+1)(-1)^m}{2} \exp\left(-\frac{(2m+1)^2\pi^2}{8} t\right), \end{aligned}$$

and $\displaystyle\int_0^\infty e^{-\alpha t} f(t) dt = 1/(\cosh\sqrt{2\alpha})$. We next record some facts concerning BRIMB. Since they are obtained via the standard arguments in the renewal theory, we omit the proof. Let $u = \displaystyle\sum_{n=1}^{\infty} f^{*n}$ where f^{*n} is the n-fold convolution of f with itself so that $\int_0^t u(s)ds = \displaystyle\sum_{n=1}^{\infty} P[\sigma_n \leq t]$. Since $E[\sigma_n - \sigma_{n-1}] = 1$ and $R_t = \max\{n \geq 0 | \sigma_n \leq t\}$, the moving boundary has the unit average speed: $\lim_{t\to\infty} R_t/t = 1$ a.s.

We denote σ_n by $\sigma(n)$ in the sequel. If $0 \leq x < t$, $y \geq 0$, and $0 \leq z \leq 1$, it holds

$$\begin{aligned} & P\left[\, t - \sigma(R_t) > x, \sigma(1+R_t) - t > y, B_t - R_t \leq z \,\right] \\ = \;& G(t,y;z) + \int_0^{t-x} u(s) G(t-s, y; z) ds. \end{aligned}$$

By the renewal theorem,

$$\lim_{t\to\infty} P\left[\, t-\sigma(R_t) > x, \sigma(1+R_t)-t > y, B_t - R_t \le z \,\right] = \int_x^\infty G(s,y;z)ds.$$

In particular, setting $x = y = 0$ we have

$$\begin{aligned}\lim_{t\to\infty} P\left[\, B_t - R_t \le z \,\right] &= \int_0^\infty G(s,0;z)ds \\ &= \frac{32}{\pi^3}\sum_{m=0}^\infty \frac{1}{(2m+1)^3}\sin\left(\frac{2m+1}{2}\pi z\right)\end{aligned}$$

for any $0 \le z \le 1$. We can deduce asymptotic independence (cf. Theorem 1 of Lalley(1984)[7]; see also p.107 of Gut(2009)[5]) between R_t and the triplet $(t-\sigma(R_t), \sigma(1+R_t)-t, B_t - R_t)$, but we omit the precise argument.

Acknowledgements. Y. Isozaki is the corresponding author and is partly supported by a grant of Japan Society for the Promotion of Science, no. 22740127.

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ブラウンラチェットのスコロホッド型方程式による特徴づけ

次のような性質を持つ確率過程を、スコロホッド型方程式により定式化し、それを「整数値移動反射壁を持つブラウンラチェット」（略称：BRIMB）と呼ぶ。この過程は、移動反射壁で上向きに反射される反射壁ブラウン運動のように振る舞い、移動反射壁より 1 だけ大きくなった瞬間に、移動反射壁がその場所に移ってくる。そのあとは、同じ方式で動き続けることにより、移動反射壁は距離 1 ずつせり上がってゆくので、ラチェットと呼ぶにふさわしい挙動をする。

ラチェットとは機械部品の名称である。逆回転を防ぐ機構が組み込まれていることが特徴であり、自転車の後輪に使われているので日常生活でなじみがある。

物理や生物の分野で、ブラウンラチェットは広く興味の対象となっている（「ファインマン物理学第 II 巻」21 章では「爪車と歯止め」と訳されている）が、本稿の定式化はそれらと違って、せり上がりの前後をまたいで成り立つ単一の方程式にもとづいている。

本稿では、そのような確率過程をブラウン運動にもとづいて構成し、確率法則の一意性を証明する。更新理論が自然に応用でき、ある種の汎関数の確率分布および、長時間での極限分布を明示できる。

キーワード：　ブラウンラチェット、ブラウン運動、スコロホッド方程式、更新理論