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# Meromorphic Differentials Restricted by Real Behavior Spaces

Dedicated to Professor Nobuyuki Suita on his sixtieth birthday

Fumio MAITANI

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**Abstract**

**This study deals with meromorphic differentials whose boundary behaviors are restricted by the real behavior spaces introduced by Shiba, but have no assumption of semiexactness, and no period condition. We show their extremal properties, and reciprocal properties of the first, second and third kind of fundamental differentials with our boundary behavior. In this context we shall formulate the Riemann-Roch and Abel type theorems on general open Riemann surfaces, which allow infinite divisors.**

**Key Words:** *Riemann surfaces; meromorphic differentials; Riemann-Roch theorem; Abel's theorem.*

## 1. Introduction

Abelian integrals theories on compact Riemann surfaces have been generalized to those on arbitrary open Riemann surfaces. For generalization the boundary behaviors of meromorphic functions and differentials under consideration need to be restricted. *L.* Ahlfors[2),3),4)], who first used distinguished differentials, and others[15),14),17),16),20)] formulated the theories in complex form as classical theories described by complex form, but their theories are, according to *R.* Accola[1)], meaningful only for Riemann surfaces with boundaries as small as those of the class $O_{KD}$. By contrast, *Y.* Kusunoki[6),7)], using semiexact canonical differentials, and others[12),22),18),19),21),11)] used real normalization, and formulated their theories in real form; meaningful for Riemann surfaces with large boundaries, and with interesting applications. For real normalization, M. Shiba[18)] introduced the notion of behavior spaces based on M.Yoshida[22)]; and showed more extended formulation of Riemann-Roch theorem for wider differential classes. We use Shiba's behavior space, simplified by Matsui[11)], but leave out the period condition in Shiba's behavior space for less restriction. We showed that the theories are established by this method even in complex form [8),9),10)]; valid for general Riemann surfaces. Although we are mainly concerned with the complex form with special meanings, it appears that the real form is more general on the point that a complex behavior space becomes a real behavior space, and has familiar applications, such as in slit mappings. Here we formulate our Riemann-Roch and Abel type theorems in real form; differing from the complex form. We use the first, second and third kind of fundamental differentials with their boundary behaviors, which play an important role in the theories. We show the reciprocal properties among them. We also point out that the restriction of boundary behavior of meromorphic differentials by our behavior space yields the extremal property of de Possel type.

## 2. Behavior spaces and fundamental differentials

Let $R$ be an arbitrary Riemann surface and $\Lambda=\Lambda(R)$ be the real Hilbert space of square integrable complex differentials whose inner product is given by

$$\langle\omega, \sigma\rangle=\text{Real part of }\iint\omega\wedge{}^*\bar{\sigma}=\mathrm{Re}(\omega, \sigma),$$

where ${}^*\sigma$ denotes the harmonic conjugate differetial of $\sigma$, and $\bar{\sigma}$ denotes the complex conjugate of $\sigma$. Let $\Lambda_{eo}$ be the completion of the class consisting of differentials of complex valued $C^\infty$-functions with compact supports, and $\Lambda_h$ be the subspace of harmonic differentials. We call a subspace $\Lambda_x(R)$ of $\Lambda_h(R)$ to be a behavior space if the orthogonal complement $\Lambda_x(R)^\perp$ of $\Lambda_x(R)$ in $\Lambda_h(R)$ is $\mathrm{i}^*\Lambda_x(R)=\{\mathrm{i}^*\omega\ ;\ \omega\in\Lambda_x(R)\}$. Let $\Gamma_h(R)=\{\omega\ ;\ \omega$ is a real differential in $\Lambda_h(R)\}$, and $\Gamma_{eo}(R)=\{\omega\ ;\ \omega$ is a real differential in $\Lambda_{eo}(R)\}$. Take a subspace $\Gamma_x(R)$ in $\Gamma_h(R)$, and denote by $\Gamma_x(R)^\perp$ the orthogonal complement in $\Gamma_h(R)$. Then $\Gamma_x(R)+\mathrm{i}^*\Gamma_x(R)^\perp$ becomes a behavior space. For a behavior space $\Lambda_x(R)$, we have the orthogonal decomposition

$$\begin{aligned}\Lambda(R)&=\Lambda_h(R)+\Lambda_{eo}(R)+{}^*\Lambda_{eo}(R)\\&=\Lambda_x(R)+\mathrm{i}^*\Lambda_x(R)+\Lambda_{eo}(R)+{}^*\Lambda_{eo}(R).\end{aligned}$$

For a behavior space $\Lambda_x(R)$, a meromorphic differential $\varphi$ is said to have $\Lambda_x$-behavior if $\varphi$ coincides with a differential $\omega\in\Lambda_x+\Lambda_{eo}$ on the outside of a compact set. Let $U$ be a parametric disk and $z$ be the local parameter usually identified with a point on the surface. For points $p$, $q\in\mathrm{U}$ there exist the $C^\infty$-differentials $\{\eta_n\}$, $\{\eta_n'\}$ which vanish on $R-U$ and satisfy

$$\eta_0=\mathrm{Re}\ \mathrm{d}\left\{T_\rho(z)\log\frac{z-q}{z-p}\right\},\quad \eta_n=-\frac{1}{n}\mathrm{Re}\ \mathrm{d}\frac{T_\rho(z)}{(z-p)^n}\quad(n\geq1),$$

$$\eta_0'=\mathrm{Im}\ \mathrm{d}\left\{T_\rho(z)\log\frac{z-q}{z-p}\right\},\quad \eta_n'=-\frac{1}{n}\mathrm{Im}\ \mathrm{d}\frac{T_\rho(z)}{(z-p)^n}\quad(n\geq1),$$

where $T_\rho(z)$ is a $C^\infty$-function, $=1$ on $U_\rho=\{z\ ;\ |z|<\rho<1\}$, $(|p|, |q|<\rho)$ and $=0$ on $\partial U=\{z\ ;\ |z|=1\}$. For a closed Jordan curve $\gamma$, take a ring domain $U_\gamma$ divided into two components by $\gamma$, and denote by $U_\gamma^+$ the component on the left hand side of $\gamma$, and by $U_\gamma^-$ the component on the right hand side of $\gamma$. There exists a real $C^\infty$-function $f_\gamma$ on $R-\gamma$ such that $f_\gamma=0$ on $R-U_\gamma^+$, and $f_\gamma=1$ in a neighborhood of $\gamma$ on $U_\gamma^+$. Then $\eta_\gamma=df_\gamma$ is a closed differential on $R$, and set $\eta_\gamma'=0$. Let $\eta$, $\eta'$ be corresponding differentials among these ones. The differential $\eta+{}^*\eta'$ vanishes on $(R-U-U_\gamma^+)\cup U_\rho$ and belongs to $\Lambda$. There is an orthogonal decomposition

$$\eta+{}^*\eta'=\omega+\tau,$$

where $\omega\in\Lambda_x+\Lambda_{eo}$, $\tau\in \mathrm{i}^*\Lambda_x+{}^*\Lambda_{eo}$. Set $\varphi'=\eta-\omega=-{}^*\eta'+\tau$ and $\varphi=\varphi'+\mathrm{i}^*\varphi'$. The $\varphi'$ becomes a closed and coclosed differential. Hence $\varphi'$ is harmonic on $R-p-q$, and $\varphi$ is analytic on $R-p-q$. The $\varphi$ coincides with $-\omega+\mathrm{i}^*\tau\in\Lambda_x+\Lambda_{eo}$ on $R-U-U_\gamma^+$ and has $\Lambda_x$-behavior. In this way we have meromorphic differentials $\psi_0=\psi_{x,p,q}$, $\psi_n=\psi_{x,n,p}$ $(n\geq 1)$ and a holomorphic differential $\psi_\gamma=\psi_{x,\gamma}$. The $\Lambda_x'=\mathrm{i}\Lambda_x$ is also a behavior space and we have $\psi_0'=\psi_{x',p,q}$, $\psi_n'=\psi_{x',n,p}$ and $\psi_\gamma'=\psi_{x',\gamma}$ These $\mathrm{i}\psi_n'$ and $\mathrm{i}\psi_\gamma'$ have $\Lambda_x$-behavior.The $\psi_0$, $\mathrm{i}\psi_0'$ are said to be the third kind of fundamental differentials with $\Lambda_x$-behavior, which have the singularities

$$\frac{dz}{z-q}-\frac{dz}{z-p},\ \frac{\mathrm{i}dz}{z-q}-\frac{\mathrm{i}dz}{z-p},\ \text{respectively.}$$

The $\psi_n$, $\mathrm{i}\psi_n'$ $(n\geq 1)$ is said to be the second kind of fundamental differentials with $\Lambda_x$-behavior, which have the singularities

$$\frac{dz}{(z-p)^{n+1}},\ \frac{\mathrm{i}dz}{(z-p)^{n+1}},\ \text{respectively.}$$

The $\psi_\gamma$, $\mathrm{i}\psi_\gamma'$ are said to be the first kind of fundamental differential with $\Lambda_x$-behavior. The finite sum of these fundamental differentials clearly becomes a meromorphic differential with $\Lambda_x$-behavior. Conversely, we have

*Proposition 1. Let $\psi$ be a meromorphic differential with $\Lambda_x$-behavior. Then there exist fundamental differentials which represent $\psi$ as follows:*

$$\psi=\sum c_{nk}\psi_{x,n,p_k}+\sum \mathrm{i}d_{nk}\psi_{x',n,p_k}+\sum c_{kj}\psi_{x,p_k,q_j}$$
$$+\sum \mathrm{i}d_{kj}\psi_{x',p_k,q_j}+\sum c_k\psi_{x,\gamma_k}+\sum \mathrm{i}d_j\psi_{x',\gamma_j}$$

Proof. We can choose a finite number of fundamental differentials and coefficients, so that

$$\varphi=\psi-\sum c_{nk}\psi_{x,n,p_k}-\sum \mathrm{i}d_{nk}\psi_{x',n,p_k}-\sum c_{kj}\psi_{x,p_k,q_j}-\sum \mathrm{i}d_{kj}\psi_{x',p_k,q_j}$$

is holomorphic. Since $\varphi$ is $\Lambda_x$-behavior, there exist a regular region G, enclosed by a finite number of analytic Jordan curves which are all dividing curves, and a differential $\omega$ in $\Lambda_x+\Lambda_{eo}$, so that $\varphi$ coincides with $\omega$ on $R-G$. Then the closed differential $\varphi-\omega$ vanishes on $R-G$, and we can choose a finite number of differentials and coefficients so that

$$\varphi_1=\varphi-\omega-\sum c_k\eta_k-\sum \mathrm{i}d_j\eta_j$$

belongs to $\Lambda_{eo}$. Then

$$\varphi_2=\varphi-\sum c_k\psi_{x,\gamma_k}-\sum \mathrm{i}d_j\psi_{x',\gamma_j}$$
$$=\omega+\varphi_1+\sum c_k(\omega_{\gamma_k}-\mathrm{i}^*\tau_{\gamma_k})+\sum \mathrm{i}d_j(\omega'_{\gamma_j}-\mathrm{i}^*\tau'_{\gamma_j}),$$

where $\omega_{\gamma_k}-\mathrm{i}^*\tau_{\gamma_k}=\eta_k-\psi_{x,\gamma_k}$, $\mathrm{i}(\omega'_{\gamma_j}-\mathrm{i}^*\tau'_{\gamma_j})=\mathrm{i}(\eta_j-\psi_{x',\gamma_j})$ belong to $\Lambda_x+\Lambda_{eo}$. Hence

$\varphi_2 = \mathrm{i}^*\varphi_2$ belongs to $(\Lambda_x+\Lambda_{eo})\cap \mathrm{i}^*(\Lambda_x+\Lambda_{eo})$ and vanishes.

## 3. Reciprocal property of fundamental differentials

The first, second and third kind of fundamental differentials have some relations among them which play a role in classical theories. We show these. Let $\Lambda_{\bar{x}} = \bar{\Lambda}_x = \{\bar{\omega}\ ;\ \omega\in\Lambda_x\}$.

*Lemma 1. Let $G$ be an open set whose every component is a simply connected region enclosed by a closed analytic Jordan curve. If meromorphic differentials $\psi$, $\varphi$ coincide with elements in $\Lambda_x+\Lambda_{eo}$, $\Lambda_{\bar{x}}+\Lambda_{eo}$ on $R-G$, respectively, then*

$$\mathrm{Im}\int_{\partial G}\Psi\varphi=0,$$

$$\mathrm{Im}\int_{\gamma'}\psi=\mathrm{Im}\int_{\partial G}\Psi_{\bar{x},\gamma'}\psi=-\mathrm{Im}\int_{\partial G}\Psi\psi_{\bar{x},\gamma'},$$

$$\mathrm{Re}\int_{\gamma'}\psi=\mathrm{Re}\int_{\partial G}\Psi_{\bar{x}',\gamma'}\psi=-\mathrm{Re}\int_{\partial G}\Psi\psi_{\bar{x}',\gamma'},$$

*where $\mathrm{d}\Psi=\psi$, $\mathrm{d}\Psi_{\bar{x}',\gamma'}=\psi_{\bar{x}',\gamma'}$ and $\partial G$ is oriented so that $G$ lies on the lefe side of $\partial G$.*

Proof. Take $\nu\in\Lambda_x+\Lambda_{eo}$, and $\mu\in\Lambda_{\bar{x}}+\Lambda_{eo}$ such that $\psi=\nu$, $\varphi=\mu$ on $R-G$. We have

$$0=\langle\psi,\mathrm{i}^*\bar{\varphi}\rangle_{R-G}=\langle\nu,\mathrm{i}^*\bar{\mu}\rangle_{R-G}$$
$$=-\langle\nu,\mathrm{i}^*\bar{\mu}\rangle_G=\mathrm{Im}\int\int_G\nu\wedge\mu=\mathrm{Im}\int_{\partial G}\Psi\varphi$$

Note that $\int_{\partial G}=\sum\int_{\partial G_i}$ is absolutely convergent, where $G_i$ is a component of $G$ and $\int_{\partial G_i}\varphi=\int_{\partial G_i}\mu=0$.

From definition there are differentials $\omega_{\bar{x},\gamma'}$, $\mathrm{i}\omega_{\bar{x}',\gamma'}\in\Lambda_{\bar{x}}+\Lambda_{eo}$, $\tau_{\bar{x},\gamma'}$, $\mathrm{i}\tau_{\bar{x}',\gamma'}\in\mathrm{i}^*\Lambda_{\bar{x}}$ so that $\psi_{\bar{x},\gamma'}=\eta_{\gamma'}-\omega_{\bar{x},\gamma'}+\mathrm{i}^*\tau_{\bar{x},\gamma'}$ and $\psi_{\bar{x}',\gamma'}=\eta_{\gamma'}-\omega_{\bar{x}',\gamma'}+\mathrm{i}^*\tau_{\bar{x}',\gamma'}$.

Then we have

$$0=\langle\psi,\mathrm{i}^*\bar{\psi}_{\bar{x},\gamma'}\rangle_{R-G}$$
$$=\langle\nu,\mathrm{i}^*(\eta_{\gamma'}-\bar{\omega}_{\bar{x},\gamma'}-\mathrm{i}^*\bar{\tau}_{\bar{x},\gamma'})\rangle_{R-G}$$
$$=\langle\nu,\mathrm{i}^*\eta_{\gamma'}\rangle_R-\langle\nu,\mathrm{i}^*\bar{\psi}_{\bar{x},\gamma'}\rangle_G$$
$$=-\mathrm{Im}\int\int_R\nu\wedge\mathrm{d}f_{\gamma'}+\mathrm{Im}\int\int_G\nu\wedge\psi_{\bar{x},\gamma'}$$
$$=\mathrm{Im}\int_{\partial(R-\gamma')}f_{\gamma'}\psi-\mathrm{Im}\int_{\partial G}\Psi_{\bar{x},\gamma'}\psi$$
$$=\mathrm{Im}\int_{\gamma'}\psi-\mathrm{Im}\int_{\partial G}\Psi_{\bar{x},\gamma'}\psi,$$

and

$$0=\langle\psi,{}^*\bar{\psi}_{\bar{x}',\gamma'}\rangle_{R-G}$$
$$=\langle\nu,{}^*(\eta_{\gamma'}-\bar{\omega}_{\bar{x}',\gamma'}-\mathrm{i}^*\bar{\tau}_{\bar{x}',\gamma'})\rangle_{R-G}$$
$$=\langle\nu,{}^*\eta_{\gamma'}\rangle_R-\langle\nu,{}^*\bar{\psi}_{\bar{x}',\gamma'}\rangle_G$$

$$=-\mathrm{Re}\int\int_R \nu\wedge \mathrm{d}f_{\gamma'}+\mathrm{Re}\int\int_G \nu\wedge\phi_{\bar{x}',\gamma'}$$
$$=\mathrm{Re}\int_{\partial(R-\gamma')} f_{\gamma'}\phi-\mathrm{Re}\int_{\partial G}\Psi_{\bar{x}',\gamma'}\phi$$
$$=\mathrm{Re}\int_{\gamma'}\phi-\mathrm{Re}\int_{\partial G}\Psi_{\bar{x}',\gamma'}\phi.$$

We have the following reciprocal properties of fundamental differentials:

*Proposition 2. For points $p$, $q\in U(z)$, $a$, $b\in U'(\zeta)$ $(U(z)\cap U'(\zeta)=\phi)$*

$$\mathrm{Re}\{\Psi_{x,p,q}(b)-\Psi_{x,p,q}(a)\}=\mathrm{Re}\{\Psi_{\bar{x},a,b}(q)-\Psi_{\bar{x},a,b}(p)\}$$
$$\mathrm{Im}\{\Psi_{x,p,q}(b)-\Psi_{x,p,q}(a)\}=\mathrm{Im}\{\Psi_{\bar{x}',a,b}(q)-\Psi_{\bar{x}'a,b}(p)\}$$

Proof. For $G=U\cup U'$ and $\phi_{x,p,q}$, $\phi_{\bar{x},a,b}$, applying Lemma 1 and residue theorem, we have

$$\mathrm{Im}\int_{\partial U'}\Psi_{x,p,q}\phi_{\bar{x},a,b}=-\mathrm{Im}\int_{\partial U}\Psi_{x,p,q}\phi_{\bar{x},a,b}=\mathrm{Im}\int_{\partial U}\Psi_{\bar{x},a,b}\phi_{x,p,q},$$

hence $\mathrm{Im}\ 2\pi\mathrm{i}\{\Psi_{x,p,q}(b)-\Psi_{x,p,q}(a)\}=\mathrm{Im}2\pi\mathrm{i}\{\Psi_{\bar{x},a,b}(q)-\Psi_{\bar{x},a,b}(p)\}$.

For $\phi_{x,p,q}$, $\mathrm{i}\phi_{\bar{x}',a,b}$

$$\mathrm{Im}\int_{\partial U'}\Psi_{x,p,q}\mathrm{i}\phi_{\bar{x}',a,b}=-\mathrm{Re}\int_{\partial U}\Psi_{x,p,q}\phi_{\bar{x}',a,b}=\mathrm{Re}\int_{\partial U}\Psi_{\bar{x}',a,b}\phi_{x,p,q},$$

hence $\mathrm{Re}\ 2\pi\mathrm{i}\{\Psi_{x,p,q}(b)-\Psi_{x,p,q}(a)\}=\mathrm{Re}\ 2\pi\mathrm{i}\{\Psi_{\bar{x}',a,b}(q)-\Psi_{\bar{x}',a,b}(p)\}$.

Similarly we have the following:

*Proposition 3. For points $p\in U(z)$, $a$, $b\in U'(\zeta)$*

$$\mathrm{Re}\{\Psi_{x,n,p}(b)-\Psi_{x,n,p}(a)\}=\mathrm{Re}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\Psi_{\bar{x},a,b}(p)\}$$
$$\mathrm{Im}\{\Psi_{x,n,p}(b)-\Psi_{x,n,p}(a)\}=\mathrm{Im}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\Psi_{\bar{x}',a,b}(p)\}$$
$$\mathrm{Re}\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\zeta^m}\{\Psi_{x,n,p}(a)\}=\mathrm{Re}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\Psi_{\bar{x},m,a}(p)\}$$
$$\mathrm{Im}\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\zeta^m}\{\Psi_{x,n,p}(a)\}=\mathrm{Im}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\Psi_{\bar{x}',m,a}(p)\}.$$

*Proposition 4. For a closed Jordan curve $\gamma'$*

$$\mathrm{Im}\int_{\gamma'}\phi_{x,n,p}=\mathrm{Re}\frac{2\pi}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\Psi_{\bar{x},\gamma'}(p)\}$$
$$\mathrm{Re}\int_{\gamma'}\phi_{x,n,p}=-\mathrm{Im}\frac{2\pi}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\Psi_{\bar{x}',\gamma'}(p)\}$$
$$\mathrm{Im}\int_{\gamma'}\phi_{x,p,q}=2\pi\mathrm{Re}\{\Psi_{\bar{x},\gamma'}(q)-\Psi_{\bar{x},\gamma'}(p)\}$$
$$\mathrm{Re}\int_{\gamma'}\phi_{x,p,q}=-2\pi\mathrm{Im}\{\Psi_{\bar{x}',\gamma'}(q)-\Psi_{\bar{x}',\gamma'}(p)\}.$$

*Proposition 5. For closed Jordan curves $\gamma$, $\gamma'$*

$$\mathrm{Im}\int_{\gamma'}\phi_{x,\gamma}=\mathrm{Im}\int_{\gamma}\phi_{\bar{x},\gamma'}$$
$$\mathrm{Re}\int_{\gamma'}\phi_{x,\gamma}=\mathrm{Re}\int_{\gamma}\phi_{\bar{x}',\gamma'}+\gamma'\times\gamma.$$

*where the intersection number $\gamma'\times\gamma=1$ if $\gamma'$ crosses $\gamma$ from the left hand side of $\gamma$ to the right.*

Proof. By calculation similar to Lemma 1

$$\begin{aligned}
0&=\langle\phi_{x,\gamma},\ \mathrm{i}^*\bar{\phi}_{\bar{x},\gamma'}\rangle_R\\
&=\langle\eta_\gamma-\omega_{x,\gamma}+\mathrm{i}^*\tau_{x,\gamma},\ \mathrm{i}^*(\eta_{\gamma'}-\bar{\omega}_{\bar{x},\gamma'}-\mathrm{i}^*\bar{\tau}_{\bar{x},\gamma'})\rangle_R\\
&=\langle\phi_{x,\gamma},\ \mathrm{i}^*\eta_{\gamma'}\rangle_R+\langle\eta_\gamma,\ \mathrm{i}^*\bar{\phi}_{\bar{x},\gamma'}\rangle_R-\langle\eta_\gamma,\ \mathrm{i}^*\eta_{\gamma'}\rangle_R\\
&=-\mathrm{Im}\int\int_R\phi_{x,\gamma}\wedge\mathrm{d}f_{\gamma'}-\mathrm{Im}\int\int_R\mathrm{d}f_\gamma\wedge\phi_{\bar{x},\gamma'}\\
&=\mathrm{Im}\int_{\partial(R-\gamma')}f_{\gamma'}\phi_{x,\gamma}-\mathrm{Im}\int_{\partial(R-\gamma)}f_\gamma\phi_{\bar{x},\gamma'}\\
&=\mathrm{Im}\int_{\gamma'}\phi_{x,\gamma}-\mathrm{Im}\int_\gamma\phi_{\bar{x},\gamma'}
\end{aligned}$$

and

$$\begin{aligned}
0&=\langle\phi_{x,\gamma},\ {}^*\bar{\phi}_{\bar{x},\gamma'}\rangle_R\\
&=\langle\eta_\gamma-\omega_{x,\gamma}+\mathrm{i}^*\tau_{x,\gamma},\ {}^*(\eta_{\gamma'}-\bar{\omega}_{\bar{x},\gamma'}-\mathrm{i}^*\bar{\tau}_{\bar{x},\gamma'})\rangle_R\\
&=\langle\phi_{x,\gamma},\ {}^*\eta_{\gamma'}\rangle_R+\langle\eta_\gamma,\ {}^*\bar{\phi}_{\bar{x},\gamma'}\rangle_R-\langle\eta_\gamma,\ {}^*\eta_{\gamma'}\rangle_R\\
&=-\mathrm{Re}\int\int_R\phi_{x,\gamma}\wedge\mathrm{d}f_{\gamma'}-\mathrm{Re}\int\int_R\mathrm{d}f_\gamma\wedge\phi_{\bar{x},\gamma'}+\mathrm{Re}\int\int_R\mathrm{d}f_\gamma\wedge\mathrm{d}f_{\gamma'}\\
&=\mathrm{Re}\int_{\partial(R-\gamma')}f_{\gamma'}\phi_{x,\gamma}-\mathrm{Re}\int_{\partial(R-\gamma)}f_\gamma\phi_{\bar{x},\gamma'}+\mathrm{Re}\int_{\partial(R-\gamma)}f_\gamma\mathrm{d}f_{\gamma'}\\
&=\mathrm{Re}\int_{\gamma'}\phi_{x,\gamma}-\mathrm{Re}\int_\gamma\phi_{\bar{x},\gamma'}-\gamma'\times\gamma.
\end{aligned}$$

## 4. Riemann-Roch type theorem

We are concerned with meromorphic functions and differentials with properties expected outside a neighborhood of poles. We establish a Riemann-Roch type theorem for such classes. Take a disjoint union $V$ of parametric disks $\{V_i\}$ which don't accumulate inside $R$. Let $\delta_p=p_1p_2p_3\cdots$, $\delta_q=q_1q_2q_3\cdots$ be integral divisors whose supports are contained in $V$, and restrictions to $V_i$ are finite divisors and set $\delta=\delta_q/\delta_p$. We consider the following vector spaces over the real number field:

$M(1/\delta_p\ ;\ \Lambda_x)=\{F\ ;\ F$ *is a multi-valued meromorphic function whose divisor is a multiple of* $\delta_p$ *and coincides with a differential in* $\Lambda_x+\Lambda_{eo}$ *on* $R-V.\}$

$S(\delta\ ;\ \Lambda_x)=\{f\in M(1/\delta_p\ ;\ \Lambda_x)\ ;\ f$ *is a single-valued meromorphic function whose divisor is a multiple of* $\delta.\}$

$D(1/\delta_q\ ;\ \bar{\Lambda}_x)=\{\varphi\ ;\ \varphi$ *is a meromorphic differential with* $\bar{\Lambda}_x$*-behavior whose divisor is a multiple of* $\delta_q.\}$

$D(1/\delta\ ;\ \bar{\Lambda}_x)=\{\varphi\ ;\ \varphi$ *is a meromorphic differential with* $\bar{\Lambda}_x$*-behavior whose divisor is a multiple of* $\delta.\}$.

We first point out the following:

*Proposition 6. For* $F\in M(1/\delta_p\ ;\ \Lambda_x)$ *and* $\varphi\in D(1/\delta_q\ ;\ \bar{\Lambda}_x)$

$$2\pi\mathrm{Re}\sum_{\{p_i\}}\mathrm{Res}F\varphi$$

$$=-\sum c_k\mathrm{Im}\int_{\gamma_k}\mathrm{d}F+\sum d_j\mathrm{Re}\int_{\gamma_j}\mathrm{d}F-\mathrm{Im}\int_{\partial V'}F\varphi,$$

*where* $\varphi=\sum c_{nk}\phi_{\bar{x},n,p_k}+\sum\mathrm{i}d_{nk}\phi_{\bar{x}',n,p_k}+\sum c_{kj}\phi_{\bar{x},p_k,q_j}$

$$+\sum\mathrm{i}d_{kj}\phi_{\bar{x}',p_k,q_j}+\sum c_k\phi_{\bar{x},\gamma_k}+\sum\mathrm{i}d_j\phi_{\bar{x}',\gamma_j}$$

*and* $V'$ *is a simply connected region which contains the poles of* $\varphi$ *and no poles of* $F$.

Proof. Let $\mathrm{d}F=\nu$, $(\nu\in\Lambda_x+\Lambda_{eo})$ on $R-V$ and on $R-V'$

$$\varphi=\sigma+\sum c_k(\tau_{\gamma_k}+\mathrm{i}^*\tau_{\gamma_k})+\sum\mathrm{i}d_j(\tau'_{\gamma_j}+\mathrm{i}^*\tau'_{\gamma_j}),$$

where $\sigma\in\bar{\Lambda}_x+\Lambda_{eo}$ and $\tau_{\gamma_k}$, $\mathrm{i}\tau'_{\gamma_j}\in\mathrm{i}^*\bar{\Lambda}_x$. It follows that

$$0=\langle\mathrm{d}F,\ \mathrm{i}^*\bar{\varphi}\rangle_{R-V\cup V'}$$

$$=\langle\nu,\ \mathrm{i}^*\bar{\sigma}+\sum c_k\mathrm{i}^*(\bar{\tau}_{\gamma_k}-\mathrm{i}^*\bar{\tau}_{\gamma_k})-\sum d_j{}^*(\bar{\tau}'_{\gamma_j}-\mathrm{i}^*\bar{\tau}'_{\gamma_j})\rangle_{R-V\cup V'}$$

$$=\langle\nu,\ \sum c_k\mathrm{i}^*\bar{\tau}_{\gamma_k}-\sum d_j{}^*\bar{\tau}'_{\gamma_j}\rangle_R+\langle\mathrm{d}F,\ \bar{\varphi}\rangle_{V\cup V'}$$

$$=\langle\sum-c_k\mathrm{i}(\mathrm{d}f_{\gamma_k}-\bar{\omega}_{\gamma_k})+\sum d_j(\mathrm{d}f_{\gamma_j}-\bar{\omega}'_{\gamma_j}),\ {}^*\nu\rangle_R+\mathrm{Im}\int_{\partial(V-V')}F\varphi+\mathrm{Im}\int_{\partial V'}F\varphi$$

$$=\sum c_k\mathrm{Im}\int_{\gamma_k}\mathrm{d}F-\sum \mathrm{d}_j\mathrm{Re}\int_{\gamma_j}\mathrm{d}F+2\pi\mathrm{Re}\sum_{\{p_i\}}\mathrm{Res}F\varphi+\mathrm{Im}\int_{\partial V'}F\varphi,$$

where we note that $\int_{\partial(V-V')}\varphi=\int_{\partial V'}\varphi=0$, hence $\int_{\partial(V-V')}F\varphi$ and $\int_{\partial V'}F\varphi$ are well defined. We get the conclusion.

We may write $\mathrm{Im}\int_{\partial V'}F\varphi=2\pi\mathrm{Re}\sum_{\{q_j\}}\mathrm{Res}F\varphi$. We can define a bilinear form on $M(1/\delta_p\ ;\ \Lambda_x)\times D(1/\delta_q\ ;\ \bar{\Lambda}_x)$ such that

$$h(F,\ \varphi)=2\pi\mathrm{Re}\sum_{\{p_i\}}\mathrm{Res}F\varphi$$

$$=-\sum c_k\mathrm{Im}\int_{\gamma_k}\mathrm{d}F+\sum \mathrm{d}_j\mathrm{Re}\int_{\gamma_j}\mathrm{d}F-2\pi\mathrm{Re}\sum_{\{q_j\}}\mathrm{Res}F\varphi.$$

We point out the following: (1) Although $F$ is multi-valued, $\mathrm{Res}_{p_i}\ F\varphi$ is well defined, because $\varphi$ is holomorphic at $p_i$. (2) Since $\varphi$ has only a finite number of singularities, $\sum_{\{q_j\}}\mathrm{Res}F\varphi$ is a finite sum.

We have the following Riemann-Roch type theorem.

*Theorem 1.*

$$\dim\frac{M(1/\delta_p\ ;\ \Lambda_x)}{S(\delta\ ;\ \Lambda_x)}=\dim\frac{D(1/\delta_q\ ;\ \bar{\Lambda}_x)}{D(1/\delta\ ;\ \bar{\Lambda}_x)}$$

*where constant functions in* $M(1/\delta_p\ ;\ \Lambda_x)$ *are regarded as zero if deg* $\delta_q\geq 1$.

Proof. If $f\in S(\delta\ ;\ \Lambda_x)$ and $\varphi\in D(1/\delta_q\ ;\ \bar{\Lambda}_x)$, then $\sum_{\{q_j\}}\mathrm{Res}f\varphi=0$ and $\int_{\gamma_j}\mathrm{d}f=0$ for every $\gamma_j$. Hence $h(f,\ \varphi)=0$, and $S(\delta\ ;\ \Lambda_x)$ is contained in the right kernel of $h$. Let $F\in M(1/\delta_p\ ;\ \Lambda_x)$, and satisfy $\mathrm{h}(F,\ \varphi)=0$ for every $\varphi\in D(1/\delta_q\ ;\ \bar{\Lambda}_x)$.

Then for $\phi_{\bar{x},\gamma_k}$, $i\phi_{\bar{x}',\gamma_k}\in D(1/\delta_q\ ;\ \bar{\Lambda}_x)$,

$$0=h(F,\ \phi_{\bar{x},\gamma_k})=-\mathrm{Im}\int_{\gamma_k}\mathrm{d}F,$$

$$0=h(F,\ i\phi_{\bar{x}',\gamma_k})=\mathrm{Re}\int_{\gamma_k}\mathrm{d}F.$$

Hence $\int_{\gamma_k}\mathrm{d}F=0$. If deg $\delta_q=0$, $F\in S(\delta\ ;\ \Lambda_x)$. Suppose deg $\delta_q\geq 1$. If $n$ is less than or equal to the multiplicity of divisor $\delta_q$ at $q_j$, then for $\phi_{\bar{x},n,q_j}$, $i\phi_{\bar{x}',n,q_j}\in D(1/\delta_q\ ;\ \bar{\Lambda}_x)$,

$$0=h(F,\ \phi_{\bar{x},n,q_j})=-2\pi\mathrm{Re}\sum_{\{q_j\}}\mathrm{Res}F\phi_{\bar{x},n,q_j}=-\frac{2\pi}{n!}\mathrm{Re}F^{(n)}(q_j),$$

and, similarly,

$$0=\mathrm{h}(F,\ i\phi_{\bar{x}',n,q_j})=-2\pi\mathrm{Re}\sum_{\{q_j\}}\mathrm{Res}F i\phi_{\bar{x}',n,q_j}=\frac{2\pi}{n!}\mathrm{Im}F^{(n)}(q_j).$$

Hence $F^{(n)}(q_j)=0$. When the support of $\delta_q$ is not a single point, we have for $\phi_{\bar{x},q_k,q_j}$, $i\phi_{\bar{x}',q_k,q_j}\in D(1/\delta_q\ ;\ \bar{\Lambda}_x)$,

$$0=h(F,\ \phi_{\bar{x},q_k,q_j})=-2\pi\mathrm{Re}\sum_{\{q_j\}}\mathrm{Res}F\phi_{\bar{x},q_k,q_j}$$
$$=2\pi\mathrm{Re}\{F(q_k)-F(q_j)\},$$

and, similarly,

$$0=h(F,\ i\phi_{\bar{x}',q_k,q_j})=-2\pi\mathrm{Re}\sum_{\{q_j\}}\mathrm{Res}F i\phi_{\bar{x}',q_k,q_j}$$
$$=-2\pi\mathrm{Im}\{F(q_k)-F(q_j)\}.$$

Hence $F(q_k)=F(q_j)$. Therefore $F$ belongs to $S(\delta\ ;\ \Lambda_x)$.

Conversely, let $\varphi\in D(1/\delta\ ;\ \bar{\Lambda}_x)$, and satisfy $h(F,\ \varphi)=0$ for every $F\in M(1/\delta_p\ ;\ \Lambda_x)$. Since $F\varphi$ is holomorphic at $\{p_i\}$,

$$h(F,\ \varphi)=2\pi\mathrm{Re}\sum_{\{p_i\}}\mathrm{Res}F\varphi=0.$$

Hence $D(1/\delta\ ;\ \bar{\Lambda}_x)$ is contained in the left kernel of $h$. Let $\varphi\in D(1/\delta_q\ ;\ \bar{\Lambda}_x)$ and satisfy $h(F,\ \varphi)=0$ for every $F\in M(1/\delta_p\ ;\ \Lambda_x)$. If $n$ is less than or equal to the multiplicity of divisor $\delta_p$ at $p_i$, then $\Psi_{x,n,p_i}\in M(1/\delta_p\ ;\ \Lambda_x)$ and

$$0=h(\Psi_{x,n,p_i},\ \varphi)=2\pi\mathrm{Re}\sum_{\{p_i\}}\mathrm{Res}\,\Psi_{x,n,p_i}\varphi=-\frac{2\pi}{n}\mathrm{Re}a_{n-1}(p_i),$$

$$0=h(i\Psi_{x',n,p_i},\ \varphi)=2\pi\mathrm{Re}\sum_{\{p_i\}}\mathrm{Res}\ i\Psi_{x',n,p_i}\varphi=\frac{2\pi}{n}\mathrm{Im}a_{n-1}(p_i),$$

where $\varphi=\sum a_m(p_i)(z-p_i)^m dz$ at $p_i$. Hence $a_{n-1}(p_i)=0$. Therefore $\varphi\in D(1/\delta\ ;\ \bar{\Lambda}_x)$ and the $D(1/\delta\ ;\ \bar{\Lambda}_x)$ is the left kernel of $h$. Thus we can get the conclusion by usual algebraic consideration.

If a holomorphic differential $\varphi$ coincides with a differential $\omega$ in $\Lambda_x+\Lambda_{eo}$ on $R-V$, then $\varphi-\omega$ vanishes on $R-V$, and exact on $R$. By Dirichlet principle $\|\varphi\|\leq\|\omega\|$. It follows that $\varphi-\omega\in\Lambda_{eo}$ and $\varphi\in\Lambda_x$. Hence $\varphi=-i^*\varphi\in\Lambda_x\cap i^*\Lambda_x$ vanishes. Therefore, if $\delta_p$ is a finite divisor, $M(1/\delta_p\ ;\ \Lambda_x)$ is generated by $\{\Psi_{x,k,p_i}\}$, $\{i\Psi_{x',k,p_i}\}$, $(1\leq k\leq\nu_i)$ and constant functions *1* and *i*, where $\nu_i$ is the multiplicity of

$\delta_p$ at $p_i$. Hence

$$\dim M(1/\delta_p\ ;\ \Lambda_x)=2\{\deg\delta_p+1-\min(\deg\delta_q,\ 1)\},$$

where constant functions in $M(1/\delta_p\ ;\ \Lambda_x)$ are regarded as zero if $\deg\delta_q\geq 1$. Thus we have

*Corollary 1. If* $\deg\delta_p<\infty$,

$$\dim S(\delta\ ;\ \Lambda_x)=2\{\deg\delta_p+1-\min(\deg\delta_q,\ 1)\}-\dim\frac{D(1/\delta_q\ ;\ \bar{\Lambda}_x)}{D(1/\delta\ ;\ \bar{\Lambda}_x)}.$$

*Particularly, if* $D(\delta'/\delta\ ;\ \bar{\Lambda}_x)=D(1/\delta\ ;\ \bar{\Lambda}_x)$ *for an integral divisor* $\delta'$,

$$\dim S(\delta/\delta'\ ;\ \Lambda_x)=\dim S(\delta\ ;\ \Lambda_x)+2\deg\delta'\geq 2\deg\delta'.$$

Further, $D(1/\delta_q\ ;\ \bar{\Lambda}_x)$ is generated by $\{\psi_{x,q_i,q_1}\}$, $\{i\psi_{\bar{x},q_i,q_1}\}$, $\{\psi_{x,k,q_k}\}$, $\{i\psi_{\bar{x},k,q_k}\}$ and $D(1\ ;\ \bar{\Lambda}_x)$ which is generated by $\{\psi_{\bar{x},r}\}$, $\{\psi_{\bar{x},r'}\}$.

*Corollary 2. When R is of finite genus g, and δ is a finite divisor,*

$$\dim S(\delta\ ;\ \Lambda_x)=\dim D(1/\delta\ ;\ \bar{\Lambda}_x)+2(1-\deg\delta)-\dim D(1\ ;\ \bar{\Lambda}_x).$$

Let $\Gamma_{hse}(R)=\{\omega\in\Gamma_h(R)\ ;\ \omega$ has no non-zero period along every dividing cycle$\}$, and $\Gamma_{hm}(R)={}^*\Gamma_{hse}(R)^{\perp}$. If a behavior space $\Lambda_x\subset\Gamma_{hse}(R)+i\Gamma_{hse}(R)$, then $i^*\Lambda_x=\Lambda_x{}^{\perp}\supset{}^*\Gamma_{hm}(R)+i^*\Gamma_{hm}(R)$, and $\Lambda_x\supset\Gamma_{hm}(R)+i\Gamma_{hm}(R)$. Hence $df_\gamma$ for dividing curve $\gamma$ belongs to $\Lambda_x\cap i\Lambda_x+\Lambda_{eo}$. It follows that $\psi_{\bar{x},r}$ and $\psi_{\bar{x}',r}$ vanish. In this case $\dim D(1\ ;\ \bar{\Lambda}_x)\leq 4g$ and $\dim S(1/p^{2g+1}\ ;\ \Lambda_x)\geq 4$. There is a non-constant meromorphic function with $\Lambda_x$-behavior whose pole is of order $2g+1$ only at $p$.

When $\Lambda_x\subset\Gamma_{hse}(R)+i\Gamma_{hse}(R)$ satisfies the follwing assumption as Shiba's period condition[18]: $-\mathrm{Im}\sum\left\{\int_{A_j}\omega\int_{B_j}\bar{\omega}-\int_{B_j}\omega\int_{A_j}\bar{\omega}\right\}=0$ for every $\omega\in\Lambda_x(R)$, we can get

$$\langle\varphi,\ \varphi\rangle_R=-\mathrm{Im}\sum\left\{\int_{A_j}\varphi\int_{B_j}\bar{\varphi}-\int_{B_j}\varphi\int_{A_j}\bar{\varphi}\right\}\ \text{for every}\ \varphi\in D(1\ ;\ \bar{\Lambda}_x),$$

where $\{A_j,\ B_j\}$ is a canonical homology basis of $R$ modulo $\partial R$. If real parts of the periods $\left\{\int_{A_j}\varphi\right\}$ and $\left\{\int_{B_j}\varphi\right\}$ vanish, the $\varphi$ also vanishes.

Then we know $\dim D(1\ ;\ \bar{\Lambda}_x)=2g$ and

$$\dim S(\delta\ ;\ \Lambda_x)=\dim D(1/\delta\ ;\ \bar{\Lambda}_x)+2(1-g-\deg\delta).$$

## 5. Abel type theorem

We now derive an Abel type theorem stating a necessary and sufficient condition for the existence of meromorphic functions with given zeros and poles, and certain restricted behavior outside of a neighborhood of their zeros and poles. We allow an infinite divisor satisfying an added condition. Take a disk $V_i'$ whose closure is contained in $V_i$ and set $V'=\sum V_i'$. Let $\delta$ be a divisor whose support is contained in $V'$ and restriction to $V_i'$ be represented as $\dfrac{q_{i1}q_{i2}\cdots q_{ik}}{p_{i1}p_{i2}\cdots p_{ik}}$. Further, assume that there exists a $C^1$-closed differential $\theta$ on $R$ such that

$$\theta=\begin{cases} d\Sigma\log\frac{z-q_{ij}}{z-p_{ij}} & \text{on } V_i' \\ 0 & \text{on } R-V, \end{cases}$$

$$\|\theta\|_{R-V'}<\infty.$$

We have the following Abel type theorem.

*Theorem 2. The following two conditions are equivalent.*

(1) *There exists a meromorphic function $f$ such that*

(i) *divisor of $f$ is $\delta$,* (ii) $d\log f=\omega$ on $R-V$, $\omega\in\Lambda_x+\Lambda_{eo}$.

(2) *There exists a chain $C=\Sigma c_{ij}$ such that for every closed Jordan curve $\gamma$ which does not meet $V$*

(i) $\mathrm{Re}\int_C\phi_{\bar{x},\gamma}\left(=\mathrm{Re}\Sigma\int_{c_{ij}}\phi_{\bar{x},\gamma}\right)$ *is an integer,*

(ii) $\mathrm{Im}\int_C\phi_{\bar{x}',\gamma}=0,$

*where $c_{ij}$ is a $C^1$-curve from $p_{ij}$ to $q_{ij}$ in $V_i$.*

Proof. If (1) is satisfied, by residue theorem

$$\int_C\phi_{\bar{x},\gamma}=\Sigma\mathrm{Res}(\Psi_{\bar{x},\gamma}d\log f)$$
$$=\frac{1}{2\pi i}\int_{\partial V}\Psi_{\bar{x},\gamma}d\log f=-\frac{1}{2\pi i}\Sigma\int_{\partial V_k}(\log f)\phi_{\bar{x},\gamma}.$$

Hence by Lemma 1,

$$\mathrm{Re}\int_C\phi_{\bar{x},\gamma}=-\frac{1}{2\pi}\mathrm{Im}\Sigma\int_{\partial V_k}(\log f)\phi_{\bar{x},\gamma}$$
$$=\mathrm{Im}\frac{1}{2\pi}\int_\gamma d\log f=\frac{1}{2\pi}\int_\gamma d\arg f,$$

and this is an integer. Similarly

$$\int_C\phi_{\bar{x}',\gamma}=\Sigma\mathrm{Res}(\Psi_{\bar{x}',\gamma}d\log f)$$
$$=\frac{1}{2\pi i}\int_{\partial V}\Psi_{\bar{x}',\gamma}d\log f=\frac{1}{2\pi}\Sigma\int_{\partial V_k}i(\log f)\phi_{\bar{x}',\gamma}.$$

Hence

$$\mathrm{Im}\int_C\phi_{\bar{x}',\gamma}=\frac{1}{2\pi}\mathrm{Re}\Sigma\int_{\partial V_k}(\log f)\phi_{\bar{x}',\gamma}$$
$$=-\mathrm{Re}\frac{1}{2\pi}\int_\gamma d\log f=-\frac{1}{2\pi}\int_\gamma d\log|f|=0.$$

By the orthogonal decomposition let represent

$$\theta-i{}^*\theta=\omega+\tau,$$

where $\omega\in\Lambda_x+\Lambda_{eo}$, $\tau\in i{}^*\Lambda_x+{}^*\Lambda_{eo}$. Set $\varphi'=\theta-\omega=i{}^*\theta+\tau$ and $\varphi=(\varphi'+i{}^*\varphi')/2$. The meromorphic differential $\varphi$ coincides with an element in $\Lambda_x+\Lambda_{eo}$ on $R-V$, and has the singularity as $\theta$.

$$\mathrm{Re}\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\varphi=\mathrm{Im}\frac{1}{2\pi}\int_{\gamma}\varphi=\mathrm{Im}\frac{1}{2\pi}\int_{\partial V}\Psi_{\bar{x},\gamma}\varphi$$
$$=\mathrm{Re}\sum\mathrm{Res}\,\Psi_{\bar{x},\gamma}\varphi=\mathrm{Re}\int_{C}\psi_{\bar{x},\gamma}.$$

Similarly

$$\mathrm{Im}\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\varphi=-\mathrm{Re}\frac{1}{2\pi}\int_{\gamma}\varphi=-\mathrm{Re}\frac{1}{2\pi}\int_{\partial V}\Psi_{\bar{x}',\gamma}\varphi$$
$$=\mathrm{Im}\sum\mathrm{Res}\,\Psi_{\bar{x}',\gamma}\varphi=\mathrm{Im}\int_{C}\psi_{\bar{x}',\gamma}.$$

By assumption (2) $\mathrm{Re}\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\varphi$ is an integer and $\mathrm{Im}\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\varphi$ vanishes. Then $\exp\left(\int\varphi\right)$ $=f$ is a meromorphic function which satisfies the condition of (1).

## 6. Extremal property of meromorphic differentials with certain boundary behavior

The meromorphic differentials treated in this paper have certain extreme characters. It is reflected in the coefficients of the Laurent development about the poles. Take real $C^1$-closed differentials $\eta_1$, $\eta_2$ such that

$$\eta_1=\begin{cases}\mathrm{Re}\ \mathrm{d}\sum_{n=-n_k}^{-1} b_{nk}(z_k)^n & V_k', \\ 0 & R-V\end{cases}$$

$$\eta_2=\begin{cases}\mathrm{Im}\ \mathrm{d}\sum_{n=-n_k}^{-1} b_{nk}(z_k)^n & V_k', \\ 0 & R-V\end{cases}$$

where $V_k'=V_{k,r_k}=\{z_k\ ;\ |z_k|<r_k<1\}$. Set $\eta=\eta_1+\mathrm{i}\eta_2$. Consider the following classes of meromorphic differentials:

$$\mathrm{Q}_x=\left\{\psi\ ;\ \mathrm{Re}(\psi-\eta)\in\Gamma_x(R)+\Gamma_{eo}(R),\ (\psi,\psi)_{R-V}\leq\mathrm{Im}\int_{\partial V}\Psi\bar{\psi}\right\},$$

$\mathrm{Q}_x{}^{\perp}=\Big\{\varphi\ ;\ \mathrm{Re}(\varphi-\eta)$ coincides with a differential in $\Gamma_x(R)^{\perp}+{}^*\Gamma_{eo}(R)$ on $R-V$ and has a finite Dirichlet integral on $V$, $(\varphi,\varphi)_{R-V}\leq\mathrm{Im}\int_{\partial V}\Phi\bar{\varphi}\Big\}$,

where $\mathrm{d}\Psi=\psi$ and $\mathrm{d}\Phi=\varphi$.

Assume that $\eta=\eta_1+\mathrm{i}\eta_2$ has a finite Dirichlet integral on $R-\cup V_k'$.

For $\eta_1+{}^*\eta_2$ there is an orthogonal decomposition

$$\eta_1+{}^*\eta_2=\omega+\tau+\omega_0+{}^*\tau_0,$$

where $\omega\in\Gamma_x$, $\tau\in\Gamma_x{}^{\perp}$ and $\omega_0$, $\tau_0\in\Gamma_{eo}$. Set $\varphi'=\eta_1-\omega-\omega_0=\tau+{}^*\tau_0-{}^*\eta_2$ and $\varphi_{\eta,x}=\varphi'+\mathrm{i}^*\varphi'$. The $\varphi_{\eta,x}$ becomes a meromorphic differential with $\Lambda_x$-behavior, where $\Lambda_x=\Gamma_x+\mathrm{i}^*\Gamma_x{}^{\perp}$.

*Lemma 2. For* $\psi\in\mathrm{Q}_x$ ($\psi=\mathrm{d}\Psi$ on $\partial V$)

$$\langle\psi,\varphi_{\eta,x}\rangle_{R-V}=-2\int_{\partial V}\mathrm{Re}\,\Psi\,\mathrm{Im}\varphi_{\eta,x}.$$

Proof. There are differentials $\sigma\in\Gamma_x$, $\sigma_0\in\Gamma_{eo}$ so that

$\mathrm{Re}(\psi-\eta)=\sigma+\sigma_0$.

It follows that

$$\begin{aligned}\langle\psi,\ \varphi_{\eta,x}\rangle_{R-V}&=\langle\psi-\eta,\ \varphi_{\eta,x}\rangle_{R-V}\\&=\langle\sigma+\sigma_0+\mathrm{i}^*(\sigma+\sigma_0),\ \tau+{}^*\tau_0-{}^*\eta_2+\mathrm{i}^*(\tau+{}^*\tau_0-{}^*\eta_2)\rangle_{R-V}\\&=\langle\sigma+\sigma_0+\mathrm{i}^*(\sigma+\sigma_0),\ \tau+{}^*\tau_0+\mathrm{i}^*(\tau+{}^*\tau_0)\rangle_{R-V}\\&=2\langle\sigma+\sigma_0,\ \tau+{}^*\tau_0\rangle_{R-V}=-2\langle\sigma+\sigma_0,\ \tau+{}^*\tau_0\rangle_V\\&=-2\mathrm{Re}\int_{\partial V}(\Psi-h)({}^*\tau-\tau_0),\end{aligned}$$

where $dh=\eta$. The $h$ is constant on every $\partial V_k$ and ${}^*\tau-\tau_0={}^*\tau-\tau_0+\eta_2=\mathrm{Im}\varphi_{\eta,x}$ on $\partial V$. Hence

$$\langle\psi,\ \varphi_{\eta,x}\rangle_{R-V}=-2\int_{\partial V}\mathrm{Re}\Psi\,\mathrm{Im}\varphi_{\eta,x}.$$

*Lemma 3. For* $\varphi\in Q_x^{\perp}(\varphi_{\eta,x}=\mathrm{d}\Phi_\eta$ on $\partial V)$

$$\langle\varphi,\ \varphi_{\eta,x}\rangle_{R-V}=-2\int_{\partial V}\mathrm{Re}\Phi_\eta\mathrm{Im}\varphi.$$

Proof. There are differentials $v\in\Gamma_x^{\perp}$, $v_0\in\Gamma_{eo}$ so that

$\mathrm{Re}(\varphi-\eta)=v+{}^*v_0$ on $R-V$.

It follows that

$$\begin{aligned}\langle\varphi,\ \varphi_{\eta,x}\rangle_{R-V}&=\langle\varphi-\eta,\ \varphi_{\eta,x}\rangle_{R-V}\\&=\langle v+{}^*v_0+\mathrm{i}^*(v+{}^*v_0),\ \eta_1-\omega-\omega_0+\mathrm{i}^*(\eta_1-\omega-\omega_0)\rangle_{R-V}\\&=\langle v+{}^*v_0+\mathrm{i}^*(v+{}^*v_0),\ -\omega-\omega_0-\mathrm{i}^*(\omega+\omega_0)\rangle_{R-V}\\&=2\langle v+{}^*v_0,\ -\omega-\omega_0\rangle_{R-V}=-2\langle v+{}^*v_0,\ -\omega-\omega_0\rangle_V\\&=-2\mathrm{Re}\int_{\partial V}(\Phi_\eta-\mathrm{h})({}^*v-v_0).\end{aligned}$$

Since ${}^*v-v_0={}^*\mathrm{Re}(\varphi-\eta)=\mathrm{Im}\varphi$ on $\partial V$,

$$\langle\varphi,\ \varphi_{\eta,x}\rangle_{R-V}=-2\int_{\partial V}\mathrm{Re}\Phi_\eta\mathrm{Im}\varphi.$$

Note that $\varphi_{\eta,x}-\eta=-\omega-\omega_0+\mathrm{i}^*(\tau+{}^*\tau_0)$ and it coincides with $\tau+{}^*\tau_0+\mathrm{i}^*(\tau+{}^*\tau_0)$ on $R-V$. Therefore $\varphi_{\eta,x}\in Q_x\cap Q_x^{\perp}$.

Let denote $\varphi\in Q_x\cup Q_x^{\perp}$ as follows

$$\varphi=\mathrm{d}\left\{\sum_{n=-n_k}^{-1}b_{nk}(z_k)^n+\sum_{n=0}^{\infty}b_{nk}(\varphi)(z_k)^n\right\}$$

on $V_k$. We have

$$\begin{aligned}&\langle\varphi,\ \varphi\rangle_{V_{k,1}-V_{k,r}}\\&=\mathrm{i}\int_{\partial(V_{k,1}-V_{k,r})}\sum_{n=-n_k}^{\infty}\sum_{m=-n_k}^{\infty}mb_{nk}\overline{b_{mk}}(z_k)^n\overline{(z_k)}^{m-1}\overline{\mathrm{d}z_k}\\&=\mathrm{i}\int_{\partial(V_{k,1}-V_{k,r})}\sum_{n=-n_k}^{\infty}\sum_{m=-n_k}^{\infty}mb_{nk}\overline{b_{mk}}(r_k)^{2n}\overline{(z_k)}^{m-n-1}\overline{\mathrm{d}z_k}\\&=2\pi\left\{\sum_{n=1}^{n_k}n|b_{-nk}|^2((r)^{-2n}-1)+\sum_{n=1}^{\infty}n|b_{nk}(\varphi)|^2(1-(r)^{2n})\right\}\\&\geqq2\pi((r)^{-2}-1)\left\{\sum_{n=1}^{n_k}n|b_{-nk}|^2+r^2\sum_{n=1}^{\infty}n|b_{nk}(\varphi)|^2\right\}\end{aligned}$$

$$\geqq 4\pi((r)^{-2}-1)r\sum_{n=1}^{n_k} n|b_{-nk}||b_{nk}(\varphi)|.$$

Hence

$$\sum_k \sum_{n=1}^{\infty} n|b_{nk}(\varphi)|^2 \leq \frac{1}{2\pi(1-\mathrm{r}^2)} \sum_k \langle \varphi, \varphi \rangle_{V_{k,1}-V_{k,r}} < \infty$$

and

$$\sum_k \sum_{n=1}^{n_k} n|b_{-nk}||b_{nk}(\varphi)| \leq \frac{r}{4\pi(1-\mathrm{r}^2)} \sum_k \langle \varphi, \varphi \rangle_{V_{k,1}-V_{k,r}} < \infty.$$

We set

$$J(\varphi) = \mathrm{Re}\sum_k \sum_{n=1}^{n_k} n b_{-nk} \mathrm{b}_{nk}(\varphi).$$

We have the following extremal property of the $\varphi_{\eta,x}$.

*Theorem 3.*

$J(\psi) \leq J(\varphi_{\eta,x})$ *for* $\psi \in Q_x$,

$J(\varphi) \geq J(\varphi_{\eta,x})$ *for* $\varphi \in Q_x{}^{\perp}$.

Proof. For $\psi$, $\varphi \in Q_x \cup Q_x{}^{\perp}$ let us denote them by $\mathrm{d}\Psi$, $\mathrm{d}\Phi$ on $V_k$. Then we have

$$\int_{\partial V_{k,r}} \Psi \overline{\mathrm{d}\Phi}$$

$$= \int_{\partial V_{k,r}} \sum_{n=-n_k}^{\infty} \sum_{m=-n_k}^{\infty} m b_{nk}(\psi) \overline{b_{mk}(\varphi)} (z_k)^n \overline{(z_k)}^{m-1} \overline{\mathrm{d}z_k}$$

$$= \int_{\partial V_{k,r}} \sum_{n=-n_k}^{\infty} \sum_{m=-n_k}^{\infty} m b_{nk}(\psi) \overline{b_{mk}(\varphi)} r^{2n} \overline{(z_k)}^{m-n-1} \overline{\mathrm{d}z_k}$$

$$= -2\pi \mathrm{i} \sum_{n=-n_k}^{\infty} n b_{nk}(\psi) \overline{b_{nk}(\varphi)} r^{2n},$$

where $b_{nk}(\psi) = b_{nk}(\varphi) = b_{nk}$ for $n<0$,

$$\int_{\partial V_{k,r}} \Psi \mathrm{d}\Phi$$

$$= \int_{\partial V_{k,r}} \sum_{n=-n_k}^{\infty} \sum_{m=-n_k}^{\infty} m b_{nk}(\psi) b_{mk}(\varphi) (z_k)^n (z_k)^{m-1} \mathrm{d}z_k$$

$$= \int_{\partial V_{k,r}} \sum_{n=-n_k}^{\infty} \sum_{m=-n_k}^{\infty} m b_{nk}(\psi) b_{mk}(\varphi) (z_k)^{n+m-1} \mathrm{d}z_k$$

$$= 2\pi \mathrm{i} \sum_{n=1}^{n_k} n b_{-nk}(b_{nk}(\varphi) - b_{nk}(\psi))$$

and

$$4\int_{\partial V_{k,r}} \mathrm{Re}\Psi \mathrm{Im}\mathrm{d}\Phi = \frac{1}{\mathrm{i}} \int_{\partial V_{k,r}} (\Psi + \overline{\Psi})(\mathrm{d}\Phi - \overline{\mathrm{d}\Phi})$$

$$= 4\pi \mathrm{Re}\left\{ \sum_{n=1}^{n_k} n b_{-nk}(b_{nk}(\varphi) - b_{nk}(\psi)) + \sum_{n=-n_k}^{\infty} n b_{nk}(\psi) \overline{b_{nk}(\varphi)} r^{2n} \right\}.$$

Hence

$$\int_{\partial V_{k,r}} \Psi \overline{\mathrm{d}\Psi} = -2\pi \mathrm{i} \sum_{n=-n_k}^{\infty} n|b_{nk}(\psi)|^2 r^{2n},$$

$$\int_{\partial V_{k,r}} \Psi \mathrm{d}\Psi = 0$$

and

$$4\int_{\partial V_{k,r}} \mathrm{Re}\Psi \mathrm{Im} \mathrm{d}\Psi = 4\pi \sum_{n=-n_k}^{\infty} n|b_{nk}(\psi)|^2 r^{2n},$$
$$= -2\mathrm{Im}\int_{\partial V_{k,r}} \Psi \overline{\mathrm{d}\Psi}.$$

Particularly

$$4\int_{\partial V_{k,r}} \mathrm{Re}(\Psi - \Phi_\eta)\mathrm{Im}\varphi_{\eta,x}$$
$$= 4\pi \mathrm{Re}\left\{\sum_{n=1}^{n_k} n b_{-nk}(b_{nk}(\varphi_{\eta,x}) - b_{nk}(\psi)) + \sum_{n=1}^{\infty} n(b_{nk}(\psi) - b_{nk}(\varphi_{\eta,x}))\overline{b_{nk}(\varphi_{\eta,x})} r^{2n}\right\},$$
$$4\int_{\partial V_{k,r}} \mathrm{Re}\Phi_\eta \mathrm{Im}(\varphi - \varphi_{\eta,x})$$
$$= 4\pi \mathrm{Re}\left\{\sum_{n=1}^{n_k} n b_{-nk}(b_{nk}(\varphi) - b_{nk}(\varphi_{\eta,x})) + \sum_{n=1}^{\infty} n b_{nk}(\varphi_{\eta,x})\overline{(b_{nk}(\varphi) - b_{nk}(\varphi_{\eta,x}))} r^{2n}\right\}.$$

For $\psi \in Q_x$, $\varphi \in Q_x{}^{\perp}$, by definition

$$\langle \psi, \psi \rangle_{R-V_r} = <\psi, \psi>_{R-V} + <\psi, \psi>_{V-V_r}$$
$$\leq \mathrm{Im}\int_{\partial V} \Psi \bar{\psi} - \mathrm{Im}\int_{\partial(V-V_r)} \Psi \bar{\psi} = \mathrm{Im}\int_{\partial V_r} \Psi \overline{\mathrm{d}\Psi}$$
$$= 2\pi \sum_k \left\{\sum_{n=1}^{n_k} n|b_{-nk}|^2 \mathrm{r}^{-2n} - \sum_{n=1}^{\infty} n|b_{nk}(\psi)|^2 r^{2n}\right\},$$
$$\langle \varphi, \varphi \rangle_{R-V_r} = <\varphi, \varphi>_{R-V} + <\varphi, \varphi>_{V-V_r}$$
$$\leq \mathrm{Im}\int_{\partial V} \Phi \bar{\varphi} - \mathrm{Im}\int_{\partial(V-V_r)} \Phi \bar{\varphi} = \mathrm{Im}\int_{\partial V_r} \Phi \overline{\mathrm{d}\Phi}$$
$$= 2\pi \sum_k \left\{\sum_{n=1}^{n_k} n|b_{-nk}|^2 r^{-2n} - \sum_{n=1}^{\infty} n|b_{nk}(\varphi)|^2 r^{2n}\right\},$$

and

$$\langle \varphi_{\eta,x}, \varphi_{\eta,x} \rangle_{R-V_r}$$
$$= 2\pi \sum_k \left\{\sum_{n=1}^{n_k} n|b_{-nk}|^2 r^{-2n} - \sum_{n=1}^{\infty} n|b_{nk}(\varphi_{\eta,x})|^2 r^{2n}\right\}.$$

Hence

$$\lim_{r\to 0}\{\langle \psi, \psi \rangle_{R-V_r} - \langle \varphi_{\eta,x}, \varphi_{\eta,x} \rangle_{R-V_r}\}$$
$$= \lim_{r\to 0} 2\pi \sum_k \sum_{n=1}^{\infty} n r^{2n}\{|b_{nk}(\varphi_{\eta,x})|^2 - |b_{nk}(\psi)|^2\} = 0.$$

Similarly $\lim_{r\to 0} \{\langle \varphi, \varphi \rangle_{R-V_r} - \langle \varphi_{\eta,x}, \varphi_{\eta,x} \rangle_{R-V_r}\} = 0.$

For $\psi \in Q_x$, $\varphi \in Q_x{}^{\perp}$, by Lemma 2 and Lemma 3

$$\langle \psi - \varphi_{\eta,x}, \varphi_{\eta,x} \rangle_{R-V_r}$$
$$= -2\pi \mathrm{Re}\sum_k \left\{\sum_{n=1}^{n_k} n b_{-nk}(b_{nk}(\varphi_{\eta,x}) - b_{nk}(\psi)) + \sum_{n=1}^{\infty} n(b_{nk}(\psi) - b_{nk}(\varphi_{\eta,x}))\overline{b_{nk}(\varphi_{\eta,x})} r^{2n})\right\}.$$

$$\langle\varphi-\varphi_{\eta,x},\ \varphi_{\eta,x}\rangle_{R-V_r}$$
$$=-2\pi\mathrm{Re}\sum_k\left\{\sum_{n=1}^{n_k} nb_{-nk}(b_{nk}(\varphi)-b_{nk}(\varphi_{\eta,x}))+\sum_{n=1}^{\infty} nb_{nk}b_{nk}(\varphi_{\eta,x})\overline{(b_{nk}(\varphi)-b_{nk}(\varphi_{\eta,x}))}r^{2n})\right\}.$$

Hence

$$\lim_{r\to 0}\langle\psi-\varphi_{\eta,x},\ \varphi_{\eta,x}\rangle_{R-V_r}=-2\pi\{J(\varphi_{\eta,x})-J(\psi)\}$$

and

$$\lim_{r\to 0}\langle\varphi-\varphi_{\eta,x},\ \varphi_{\eta,x}\rangle_{R-V_r}=-2\pi\{J(\varphi)-J(\varphi_{\eta,x})\}.$$

It follows that

$$0\le\|\psi-\varphi_{\eta,x}\|^2=\lim_{r\to 0}\langle\psi-\varphi_{\eta,x},\ \psi-\varphi_{\eta,x}\rangle_{R-V_r}$$
$$=\lim_{r\to 0}\{\langle\psi,\ \psi\rangle_{R-V_r}-\langle\varphi_{\eta,x},\ \varphi_{\eta,x}\rangle_{R-V_r}-2\langle\psi-\varphi_{\eta,x},\ \varphi_{\eta,x}\rangle_{R-V_r}\}$$
$$=4\pi\{J(\varphi_{\eta,x})-J(\psi)\},$$

and

$$0\le\|\varphi-\varphi_{\eta,x}\|^2=4\pi\{J(\varphi)-J(\psi_{\eta,x})\}.$$

Thus the statement follows.

*Department of Mechanical and System Engineering,*
*Faculty of Engineering and Design,*
*Kyoto Institute of Technology.*
*Matsugasaki, Sakyo-ku, Kyoto 606*

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