Abelian Teichmüller disks restricted by behavior

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

By

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Abstract

We shall investigate quasiconformal deformations of arbitrary Riemann surfaces whose Riemannian metrics are given by Beltrami differentials with complex parameters in the unit disk, where the Beltrami differentials are represented by certain holomorphic differentials with restricted boundary behavior. They are so called Abelian Teichmüller disks. For the sake of discussion on general open Riemann surfaces, we consider two types of boundary behavior. The one is based on Dirichlet finite harmonic differentials with integral periods[8] and the other is based on holomorphic differentials with normal behavior[4]. By using the arguments of F. Gardiner[1] and I. Kra[3] on punctured Riemann surfaces, we shall discuss extremal quasiconformal mappings in Torelli space, variational formulas of the norms of our restricted holomorphic differentials and Torelli's theorem on our Abelian Teichmüller disks of an arbitrary Riemann surface.

1. Introduction

Let S be an open Riemann surface and $M(S)$ be the set of Beltrami differentials:

$$\left\{\mu=\underline{\mu}\frac{d\bar{z}}{dz};\ \underline{\mu} \text{ is measurable and } \|\mu\|_\infty=\text{esssup}\,|\underline{\mu}|<1\right\}$$

on S. From $\mu\in M(S)$ we get another Riemann surface S_μ with the Riemannian metric $ds=\lambda(z)|dz+\underline{\mu}(z)d\bar{z}|$. For $\mu, \nu\in M(S)$, μ and ν is R-equivalent if there exists a conformal mapping between S_μ and S_ν and T-equivalent (resp. L-equivalent) if there exists a homeomorphism which is homotopic (resp. homologic) to the identity mapping and it is regarded as a conformal mapping between S_μ and S_ν. Let $\tilde{R}(S)$, $\tilde{T}(S)$ and $\tilde{L}(S)$ be the quotient space of $M(S)$ by R-equivalent, T-equivalent and L-equivalent, respectively. They are called Riemann space, Teichmüller space and Torelli space. In the case of a compact Riemann surface S, using the Teichmüller theory

$$\tilde{T}(S)=\left\{t\frac{\bar{\omega}}{|\omega|}:\ \omega \text{ is a holomorphic quadratic differential and } |t|<1\right\}.$$

Key words: Riemann surface; quasiconformal mapping; Beltrami differential; Teichmüller disk; Torelli's theorem.

For a open Riemann surface S, let

$$T(S)=\left\{t\frac{\bar{\omega}}{|\omega|}:\ \omega \text{ is an integrable holomorphic quadratic differential and } |t|<1\right\}.$$

For an integrable holomorphic quadratic differential ω, we call $K_\omega=\left\{t\frac{\bar{\omega}}{|\omega|}: |t|<1\right\}$ a Teichmüller disk, and for a square integrable holomorphic differential θ we call K_{θ^2} an abelian Teichimüller disk. We are concerned with the class:

$$T_a(S)=\left\{t\frac{\bar{\theta}}{\theta}:\ \theta \text{ is a square integrable holomorphic differential and } |t|<1\right\}.$$

Here we shall study certain abelian Teichmüller disks. For the investigation on an open Riemann surface S, we need to restrict the boundary behavior of holomorphic differentials. Firstly, we consider the restriction by Taniguchi's Dirichlet finite harmonic differentials with integral periods. We shall give an inequality[1] with respect to the restricted holomorphic differentials and Beltrami differentials of quasiconformal self mappings which fix the homology. It follows Reich-Strebel's fundamental inequality[6] in our circumstances. We also give certain variational formulas of the norm of the restricted holomorphic differential. Secondly, by using normal behavior[4] and I. Kra's arguments[3], we show that Riemann's period matrices of different points in the restricted Abelian Teichmüller disks are different.

2. Dirichlet finite harmonic differentials with integral periods

We use the following class of differentials. Let Λ be the real Hilbert space of square integrable complex differentials whose inner product is given by

$$\langle\omega,\sigma\rangle=\text{Real part of }\iint\omega\wedge{}^*\bar{\sigma}=\mathrm{Re}\,(\omega,\sigma),$$

where ${}^*\sigma$ denotes the harmonic conjugate differential of σ and $\bar{\sigma}$ denotes the complex conjugate of σ. Let Λ_h be the subspace of harmonic differentials and Λ_{eo} the completion of the class which consists of the differentials of complex valued C^∞-functions with compact supports. Write the subspace of real harmonic differentials as Γ_h and for a subspace Γ_x of Γ_h denote the orthogonal complement in Γ_h by $\Gamma_x^\perp$, the harmonic conjugate by ${}^*\Gamma_x$. Dr. M. Taniguchi introduced the class Γ_{hi} of square integrable harmonic differentials with integral periods *i.e.*, $\int_\gamma\sigma$ is integral for a closed curve γ and $\sigma\in\Gamma_{hi}$.[8] For $\sigma\in\Gamma_{hi}$, $S_\sigma(p,p_0)=\exp 2\pi i\int_{p_0}^{p}\sigma$ becomes a function on the Riemann surface and takes values in the unit circle. It is called a circular function. Remark that $S_\sigma(p,p_0)-S_\sigma(p,p_1)$ is constant and we abbreviate it as $S_\sigma(p)$. Since $S_\sigma(p)$ has a finite Dirichlet integral, it has a continuous extension to Royden's compactification S*. The image $E(\sigma)$ by S_σ of the harmonic boundary is compact in the unit circle. Consider a subclass of Γ_{hi}: $\Gamma_{hic}=\{\sigma\in\Gamma_{hi}$: The $E(\sigma)$ is of linear measure zero$\}$. Set $\Lambda_{aic}=\{\theta_\sigma=-{}^*\sigma+i\sigma:\ \sigma\in\Gamma_{hic}\}$. Now a horizontal trajectory γ of a meromorphic quadratic differential ϕ is defined as a maximal curve along which ϕ is positive. A trajectory γ is called critical if γ tends to a zero or a pole of ϕ in either direction, and regular, if otherwise. We set $E_0=\{S_\sigma(p)$: a point p is a zero of $\theta_\sigma\}$. This is a countable set. For every $\exp 2\pi i a \notin E_0$, $S_\sigma^{-1}(\exp 2\pi i a)$ consists of analytic curves which are regular trajectories of θ_σ^2, because $\int^p \sigma = a \bmod Z$ and $\theta_\sigma^2=(-{}^*\sigma)^2$ on $S_\sigma^{-1}(\exp 2\pi i a)$. Let $L(a)$ be the set $\cup C_j(a)$ of all trajectories $C_j(a)$ of θ_σ^2 in $S_\sigma^{-1}(\exp 2\pi i a)$ and $C_j(a)$ be oriented so that $\int_{C_j(a)} \theta_\sigma > 0$. The set $L(a)$ ($\exp 2\pi i a \notin E_0$) is locally connected.[8] Define $W_\sigma(a)=\sum_j \int_{C_j(a)} \theta_\sigma$ and $W_\sigma(a)=0$ if $L(a)$ is the empty set. The complement of $E(\sigma)$ in the unit circle is an open set, hence it is a union $\cup I_i(\sigma)$ of open arcs $I_i(\sigma)$. Let the length of $I_i(\sigma)$ be $2\pi m_i(\sigma)$. Dr. M. Taniguchi showed that

Theorem A.[8] *For $\theta_\sigma(\neq 0)\in \Lambda_{aic}$, the complement of all compact regular trajectories of θ_σ^2 is a set of 2-dimensional measure zero. If $a, \beta \in I_i(\sigma)$, then $W_\sigma(a)=W_\sigma(\beta)$ and denote it as $W_i(\sigma)$. It holds that*

$$\|\theta_\sigma\|^2 = 2\sum m_i(\sigma) W_i(\sigma) = 2\int_0^1 W_\sigma(a)\mathrm{d}a$$

From this result it follows that

Lemma 1. *Let $\theta_\sigma \in \Lambda_{aic}$ and ψ be a quadratic differential such that for every compact regular trajectory $C_j(a)$ of θ_σ^2*

$$\int_{C_j(a)} |\sqrt{\psi}| \geqq \int_{C_j(a)} \theta_\sigma .$$

Then

$$\|\theta_\sigma\|^2 \leqq \iint_S |\theta_\sigma \wedge \sqrt{\overline{\psi}}|.$$

Proof. For $z_0 \in C_j(a)$, write $\int_{z_0}^z \theta_\sigma + ia = \xi + i\eta = \zeta$ and

$$\int_{C_j(a)} |\sqrt{\psi}| = \int_{C_j(a)} |\sqrt{\underline{\psi}(\xi+ia)}\, \mathrm{d}\zeta| \geqq \int_{C_j(a)} \mathrm{d}\xi, \text{ where } \psi(\zeta)=\underline{\psi}(\zeta)\mathrm{d}\zeta^2.$$

We have

$$\iint |\theta_\sigma \wedge \sqrt{\overline{\psi}}\,| = \iint |\mathrm{d}\zeta \wedge \sqrt{\overline{\underline{\psi}(\zeta)}}\, \mathrm{d}\bar{\zeta}|$$
$$= 2\int_0^1 \int_{L(\eta)} |\sqrt{\overline{\underline{\psi}(\xi+i\eta)}}| \mathrm{d}\xi \mathrm{d}\eta \geqq 2\int_0^1 \int_{L(\eta)} \mathrm{d}\xi \mathrm{d}\eta = 2\int_0^1 W_\sigma(\eta)\mathrm{d}\eta = \|\theta_\sigma\|^2.$$

3. Extremal quasiconformal mappings

Let f be a quasiconformal mapping on S which satisfies that every Jordan curve γ is homologous to $f(\gamma)$. Denote the Beltrami differential of f by $\mu(z)=\dfrac{f_{\bar{z}}\mathrm{d}\bar{z}}{f_z \mathrm{d}z}$. It follows that

Theorem 1.[1] *For $\theta_\sigma \in \Lambda_{aic}$*

$$\|\theta_\sigma\|^2 \leqq \iint \frac{|1+\mu\theta_\sigma/\bar{\theta}_\sigma|^2}{1-|\mu|^2} |\theta_\sigma \wedge \bar{\theta}_\sigma|.$$

Proof. Let a quadratic differential ψ be defined by

$$\psi=\{(1+\mu\theta_\sigma/\bar{\theta}_\sigma)(\theta_\sigma\circ f+i^*(\theta_\sigma\circ f))/2\}^2.$$

For a compact regular trajectory C_j of θ_σ^2, choosing the parameter ζ as in the proof of Lemma 1, write

$$|\sqrt{\psi}|=|(1+f_{\bar{\zeta}}/f_\zeta)\underline{\theta}_\sigma(f)f_\zeta d\zeta| \qquad (\theta_\sigma=\underline{\theta}_\sigma(z)dz)$$
$$=|\underline{\theta}_\sigma(f)f_\zeta d\zeta+\underline{\theta}_\sigma(f)f_{\bar{\zeta}}d\zeta|.$$

Since $d\zeta=d\bar{\zeta}$ along C_j, we have

$$\int_{C_j}|\sqrt{\psi}|=\int_{C_j}|\underline{\theta}_\sigma(f)f_\zeta d\zeta+\underline{\theta}_\sigma(f)f_{\bar{\zeta}}d\bar{\zeta}|$$
$$=\int_{C_j}|\underline{\theta}_\sigma(f)df|=\int_{f(C_j)}|\theta_\sigma|\geqq \mathrm{Re}\int_{f(C_j)}\theta_\sigma=\int_{C_j}\theta_\sigma.$$

By Lemma 1, it follows that

$$\|\theta_\sigma\|^2\leqq\iint|\theta_\sigma\wedge\sqrt{\bar{\psi}}|$$
$$=\iint|\underline{\theta}_\sigma(f)(1+\mu\theta_\sigma/\bar{\theta}_\sigma)f_\zeta d\zeta\wedge d\bar{\zeta}|$$
$$\leqq\left\{\iint|\underline{\theta}_\sigma(f)^2 f_\zeta{}^2(1-|\mu|^2)d\zeta\wedge d\bar{\zeta}|\right\}^{1/2}\left\{\iint\frac{|1+\mu\theta_\sigma/\bar{\theta}_\sigma|^2}{1-|\mu|^2}|d\zeta\wedge d\bar{\zeta}|\right\}^{1/2}$$
$$=\|\theta_\sigma\|\left\{\iint\frac{|1+\mu\theta_\sigma/\bar{\theta}_\sigma|^2}{1-|\mu|^2}|d\zeta\wedge d\bar{\zeta}|\right\}^{1/2}.$$

Hence

$$\|\theta_\sigma\|^2\leqq\iint\frac{|1+\mu\theta_\sigma/\bar{\theta}_\sigma|^2}{1-|\mu|^2}|\theta_\sigma\wedge\bar{\theta}_\sigma|.$$

Let f_i be a quasiconformal mapping from S_i to S_{i+1} ($i=0, 1$) and μ_i be the Beltrami differential of f_i. Denote the Beltrami differential of f_i^{-1} by ν_i and the Beltrami differential of $f_1\circ f_0$ by τ.

Lemma 2.

$$\tau=\frac{\mu_0-\mu_1\mu_0/\nu_0}{1-\mu_1\bar{\nu}_0}.$$

Proof. Take parameters z, w, ζ on S_0, S_1, S_2, respectively, and write

$$\mu_0=\frac{w_{\bar{z}}d\bar{z}}{w_z dz},\ \mu_1=\frac{\zeta_{\bar{w}}d\bar{w}}{\zeta_w dw},\ \nu_0=\frac{z_{\bar{w}}d\bar{w}}{z_w dw},\ \tau=\frac{\zeta_{\bar{z}}d\bar{z}}{\zeta_z dz}.$$

Since $\zeta_{\bar{z}}=\zeta_w w_{\bar{z}}+\zeta_{\bar{w}}\bar{w}_{\bar{z}}$, $\zeta_z=\zeta_w w_z+\zeta_{\bar{w}}\bar{w}_z$, we have

$$\tau=\frac{\zeta_w w_{\bar{z}}+\zeta_{\bar{w}}\bar{w}_{\bar{z}}}{\zeta_w w_z+\zeta_{\bar{w}}\bar{w}_z}\frac{w_z}{w_{\bar{z}}}\mu_0=\frac{\mu_0+\zeta_{\bar{w}}\bar{w}_{\bar{z}}\mu_0/\zeta_w w_{\bar{z}}}{1+\zeta_{\bar{w}}\bar{w}_z/\zeta_w w_z}.$$

From $z_w w_{\bar{z}}+z_{\bar{w}}\bar{w}_{\bar{z}}=0$, it follows that

$$\nu_0=-\frac{w_{\bar{z}}d\bar{w}}{\bar{w}_{\bar{z}}dw},\ \bar{\nu}_0=-\frac{\bar{w}_z dw}{w_z d\bar{w}}\ \text{and}\ \frac{\mu_1}{\nu_0}=-\frac{\zeta_{\bar{w}}}{\zeta_w}\frac{\bar{w}_{\bar{z}}}{w_{\bar{z}}}.$$

Thus

$$\tau=\frac{\mu_0-\mu_1\mu_0/\nu_0}{1-\mu_1\bar{\nu}_0}.\quad \text{(cf. If } \mu_0=\nu_0=0, \text{ let } \mu_0/\nu_0=-1.)$$

Let $S_2=S_0$ and $f_1\circ f_0=h$ satisfy that every Jordan curve γ is homologous to $h(\gamma)$. We have a fundamental inequality of a Reich and Strebel type.

Theorem 2.[2),6),7)] *For* $\theta_\sigma\in\Lambda_{aic}(S_0)$

$$\|\theta_\sigma\|^2\leqq\iint\frac{|1+\mu_0\theta_\sigma/\bar{\theta}_\sigma|^2}{(1-|\mu_0|^2)(1-|\mu_1|^2)}\left|1-\frac{\mu_0\theta_\sigma}{\nu_0\bar{\theta}_\sigma}\mu_1\frac{1+\bar{\mu}_0\bar{\theta}_\sigma/\theta_\sigma}{1+\mu_0\theta_\sigma/\bar{\theta}_\sigma}\right|^2|\theta_\sigma\wedge\bar{\theta}_\sigma|.$$

Proof. By Theorem 1

$$\|\theta_\sigma\|^2\leqq\iint\frac{|1+\tau\theta_\sigma/\bar{\theta}_\sigma|^2}{1-|\tau|^2}|\theta_\sigma\wedge\bar{\theta}_\sigma|.$$

From Lemma 2

$$1+\tau\frac{\theta_\sigma}{\bar{\theta}_\sigma}=1+\frac{1-\mu_1/\nu_0}{1-\mu_1\bar{\nu}_0}\cdot\frac{\theta_\sigma}{\bar{\theta}_\sigma}\mu_0$$

$$=\frac{1}{1-\mu_1\bar{\nu}_0}\left(1+\mu_0\frac{\theta_\sigma}{\bar{\theta}_\sigma}\right)\left(1-\frac{\mu_0\theta_\sigma}{\nu_0\bar{\theta}_\sigma}\mu_1\frac{1+|\nu_0|^2\bar{\theta}_\sigma/\mu_0\theta_\sigma}{1+\mu_0\theta_\sigma/\bar{\theta}_\sigma}\right),$$

$$1-|\tau|^2=\frac{1}{|1-\mu_1\bar{\nu}_0|^2}\{|1-\mu_1\bar{\nu}_0|^2-|(1-\mu_1/\nu_0)\mu_0|^2\}$$

$$=\frac{1}{|1-\mu_1\bar{\nu}_0|^2}\left\{1-|\mu_0|^2+|\mu_1|^2\left(|\nu_0|^2-\left|\frac{\mu_0}{\nu_0}\right|^2\right)-2\,\mathrm{Re}\,\mu_1\left(\bar{\nu}_0-\frac{|\mu_0|^2}{\nu_0}\right)\right\}.$$

Since $|\nu_0|=|\mu_0|$,

$$1+\tau\frac{\theta_\sigma}{\bar{\theta}_\sigma}=\frac{1}{1-\mu_1\bar{\nu}_0}\left(1+\mu_0\frac{\theta_\sigma}{\bar{\theta}_\sigma}\right)\left(1-\frac{\mu_0\theta_\sigma}{\nu_0\bar{\theta}_\sigma}\mu_1\frac{1+\bar{\mu}_0\bar{\theta}_\sigma/\theta_\sigma}{1+\mu_0\theta_\sigma/\bar{\theta}_\sigma}\right),$$

$$1-|\tau|^2=\frac{1}{|1-\mu_1\bar{\nu}_0|^2}(1-|\mu_0|^2)(1-|\mu_1|^2).$$

Thus the inequality follows.

Let g be a quasiconformal mapping from S_0 to S_1 and

$F=\{f: f$ *is a quasiconformal mapping from* S_0 *to* S_1 *such that* $f^{-1}\circ g(\gamma)$ *is homologous to* γ *for every Jordan curve* $\gamma\}$.

Denote the Beltrami differential of $f\in F$ by μ_f. Consider the following extremal problem:

$$\inf\{\|\mu_f\|: f\in F\}=k.$$

When $\|\mu_f\|$ attains the infimum k, we call f an extremal quasiconformal mapping among F.

Corollary 1.[2),6),7)] *For every* $\theta_\sigma\in\Lambda_{aic}$ *and* $f\in F$,

$$\frac{1-k}{1+k}\|\theta_\sigma\|^2\leqq\iint\frac{|1+\mu_f\theta_\sigma/\bar{\theta}_\sigma|^2}{1-|\mu_f|^2}|\theta_\sigma\wedge\bar{\theta}_\sigma|$$

Proof. For any $\varepsilon>0$, we can choose a quasiconformal mapping f_1 from S_1 to S_0 such that $f_1^{-1}\in F$ and $k\leqq\|\mu_{f_1}\|=k_1<k+\varepsilon$.

From Theorem 2

$$\|\theta_\sigma\|^2\leqq\frac{1+k_1}{1-k_1}\iint\frac{|1+\mu_f\theta_\sigma/\bar{\theta}_\sigma|^2}{1-|\mu_f|^2}|\theta_\sigma\wedge\bar{\theta}_\sigma|$$

This gives the inequality.

Corollary 2. *Let the Beltrami differential of* $f_0 \in F$ *be* $-k_0\bar{\theta}_\sigma/\theta_\sigma$, *where* k_0 *is positive. Then* $k_0=k$ *and* f_0 *is a unique extremal quasiconformal mapping among* F.

Proof. For $\mu_0=-k_0\bar{\theta}_\sigma/\theta_\sigma$ and $\|\mu_1\|=k_1$, applying Theorem 2, we have

$$\|\theta_\sigma\|^2 \leqq \frac{1-k_0}{1+k_0}\iint \frac{|1+k_0\mu_1/\nu_0|^2}{1-|\mu_1|^2}|\theta_\sigma\wedge\bar{\theta}_\sigma| \leqq \frac{1-k_0}{1+k_0}\,\frac{1+k_1}{1-k_1}\|\theta_\sigma\|^2.$$

Hence $\frac{1-k_0}{1+k_0}\frac{1+k_1}{1-k_1}\geqq 1$ and $k_1 \geqq k_0$. Further if $k_1=k_0$, then $\mu_1=\nu_0$ and $f_1^{-1}=f_0$. This proves the statement.

4. Variational formulas

Let f_μ be the quasiconformal mapping from S to S_μ whose Beltrami coefficient is $\underline{\mu}=(f_\mu)_{\bar{z}}/(f_\mu)_z$. Take a $\mu\in M(S)$ and set $T(S,\mu)=\{t\mu\colon |t|<1\}$. We shall write as $(S_t;f_t)$ instead of $(S_{t\mu};f_{t\mu})$. Now for $\Gamma_x\subset\Gamma_h$, set $\Lambda_x=\Gamma_x+\mathrm{i}^*\Gamma_x^\perp$. Take a meromorphic differential ϕ^t on S_t ($S_0=S$) such that $\phi^t\circ f_t-\phi^0\in\Lambda_x(S)+\Lambda_{eo}(S)$. We have shown that there exist differentials ϕ_u^t, ϕ_v^t in $\Lambda_x(S_t)+\Lambda_{eo}(S_t)$ such that

$$\begin{aligned}\phi_u^t-\mathrm{i}^*\phi_u^t&=-\mathrm{i}(\phi_v^t-\mathrm{i}^*\phi_v^t)\\&=\phi^t\frac{2\underline{\mu}(z(\zeta))}{1-|t\underline{\mu}(z(\zeta))|^2}\,\frac{\zeta_z}{\bar{\zeta}_{\bar{z}}}\,\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta}.^{5)}\end{aligned}$$

Set $\phi_{\bar{t}}^t=(\phi_u^t+\mathrm{i}\phi_v^t)/2$ and $\phi_t^t=(\phi_u^t-\mathrm{i}\phi_v^t)/2$. The $\phi_{\bar{t}}^t$ becomes a holomorphic differential. We have also shown that

Theorem B.[5] *Let meromorphic differentials* ϕ^t, ψ^t *satisfy that the poles of* ϕ^0, ψ^0 *do not meet the support of* μ *and* $\phi^t\circ f_t-\phi^0$, $\psi^t\circ f_t-\psi^0\in\Lambda_x(S)+\Lambda_{eo}(S)$. *Then*

$$\begin{aligned}\frac{\partial}{\partial t}\langle\phi^t\circ f_t-\phi^0,\bar{\psi}^0\rangle&=\frac{1}{2}(\phi_t^t,\bar{\psi}^t)\\&=\frac{\mathrm{i}}{2}\iint\underline{\phi}^t\underline{\psi}^t\underline{\mu}\zeta_z{}^2\mathrm{d}z\mathrm{d}\bar{z}\qquad(\phi^t=\underline{\phi}^t\mathrm{d}\zeta,\ \psi^t=\underline{\psi}^t\mathrm{d}\zeta),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\bar{t}\partial t}\langle\phi^t\circ f_t-\phi^0,\bar{\psi}^0\rangle&=\frac{1}{2}\{(\phi_{\bar{t}}^t,\bar{\psi}_t^t)+(\psi_{\bar{t}}^t,\bar{\phi}_t^t)\}\\&=\frac{\mathrm{i}}{2}\iint(\underline{\phi}_{\bar{t}}^t\underline{\psi}^t+\underline{\psi}_{\bar{t}}^t\underline{\phi}^t)\underline{\mu}\zeta_z^2\mathrm{d}z\mathrm{d}\bar{z}\quad(\phi_{\bar{t}}^t=\underline{\phi}_{\bar{t}}^t\mathrm{d}\zeta,\ \psi_{\bar{t}}^t=\underline{\psi}_{\bar{t}}^t\mathrm{d}\zeta)\end{aligned}$$

Further, when ϕ^t, ψ^t are holomorphic and $\Lambda_x=\Gamma_h$,

$$\frac{\partial}{\partial t}\langle\phi^t\circ f_t-\phi^0,\bar{\psi}^0\rangle=\frac{1}{2}(\psi^t,\phi_{\bar{t}}^t),$$

$$\frac{\partial^2}{\partial\bar{t}\partial t}\langle\phi^t\circ f_t-\phi^0,\bar{\psi}^0\rangle=\langle\phi_{\bar{t}}^t,\psi_{\bar{t}}^t\rangle.$$

Particulary,

$$\frac{\partial}{\partial t}\langle\phi^t,\psi^t\rangle=\frac{1}{2}\{(\phi^t,\psi_{\bar{t}}^t)+(\psi^t,\phi_{\bar{t}}^t)\},$$

$$\frac{\partial^2}{\partial\bar{t}\partial t}\langle\phi^t,\psi^t\rangle=2\langle\phi_{\bar{t}}^t,\psi_{\bar{t}}^t\rangle.$$

Since pull back of differential by quasiconformal mapping preserves the period and the finiteness of the norm, for $\sigma\in\Gamma_{hic}(S)$ $\sigma\circ f_t^{-1}$ has integral periods and belongs to $\Gamma(S_t)$. Consider the orthogonal decomposition of $\sigma\circ f_t^{-1}$:

$$\sigma\circ f_t^{-1}=\sigma_t+\mathrm{d}P_0,$$

where $\sigma_t\in\Gamma_{hi}(S_t)$ and P_0 is a Dirichlet potential. The P_0 vanishes on the harmonic boundary of Royden's compactification of S_t^*. Hence $E(\sigma)=E(\sigma_t)$ for $S_\sigma(p,p_0)$, $S_{\sigma_t}(p,f_t(p_0))$ and $E(\sigma_t)$ is of linear measure zero. Thus $\sigma_t\in\Gamma_{hic}(S_t)$ and $\theta^t=-{}^*\sigma_t+\mathrm{i}\sigma_t\in\Lambda_{aic}(S_t)$. The $\theta^t\circ f_t-\theta_\sigma$ belongs to $\Gamma_h+\Lambda_{eo}$, because

$$\theta^t\circ f_t-\theta_\sigma=-{}^*\sigma_t\circ f_t+{}^*\sigma-\mathrm{id}(P_0\circ f_t).$$

By Theorem B we have the following.

Theorem 3.[1]

$$\frac{\partial}{\partial t}\log\|\theta^t\|=\frac{\mathrm{i}}{2\|\theta^t\|^2}\iint\underline{\theta}^{t2}\mu\zeta_z^2\mathrm{d}z\mathrm{d}\bar{z}\quad(\theta^t=\underline{\theta}^t\mathrm{d}\zeta)$$

$$=\frac{1}{2\|\theta^t\|^2}(\theta^t,\ \theta^t_{\bar{t}}),$$

$$\frac{\partial^2}{\partial\bar{t}\partial t}\log\|\theta^t\|=\frac{1}{\|\theta^t\|^2}\left\{\|\theta^t_{\bar{t}}\|^2-\frac{1}{2\|\theta^t\|^2}|(\theta^t,\ \theta^t_{\bar{t}})|^2\right\}\geqq\frac{\|\theta^t_{\bar{t}}\|^2}{2\|\theta^t\|^2}\geqq0.$$

Proof. We have

$$\frac{\partial}{\partial t}\log\|\theta^t\|^2=\frac{1}{\|\theta^t\|^2}\frac{\partial}{\partial t}\|\theta^t\|^2=\frac{1}{\|\theta^t\|^2}(\theta^t,\ \theta^t_{\bar{t}})=\frac{\mathrm{i}}{\|\theta^t\|^2}\iint\underline{\theta}^{t2}\mu\zeta_z^2\mathrm{d}z\mathrm{d}\bar{z},$$

$$\frac{\partial^2}{\partial\bar{t}\partial t}\log\|\theta^t\|^2=\frac{-1}{\|\theta^t\|^4}\left|\frac{\partial}{\partial t}\|\theta^t\|^2\right|^2+\frac{1}{\|\theta^t\|^2}\frac{\partial}{\partial\bar{t}\partial t}\|\theta^t\|^2$$

$$=\frac{1}{\|\theta^t\|^2}\left\{2\|\theta^t_{\bar{t}}\|^2-\frac{1}{\|\theta^t\|^2}|(\theta^t,\ \theta^t_{\bar{t}})|^2\right\}\geqq\frac{\|\theta^t_{\bar{t}}\|^2}{\|\theta^t\|^2}\geqq0.$$

H. Yamaguchi considered, in his investigation[9] of analytic variations of pseudo convex domains, the case which Robin constants vary harmonically. We also consider that $\log\|\theta^t\|$ varies harmonically, then by Theorem 3 we have $\theta^t_{\bar{t}}=0$ and $\|\theta^t\|$ is constant. When S is a compact bordered Riemann surface with n genus and m boundary components. For a canonical homology basis $\{A_j,\ B_j,\ C_i\}$, take periods reproducing harmonic differentials $\sigma(A_j)$, $\sigma(B_j)$, $j=1\ldots n$, and $\sigma(C_i)$, $i=1\ldots m-1$ such that

$$\iint\sigma(A_j)\wedge\omega=\int_{A_j}\omega,\quad\iint\sigma(B_j)\wedge\omega=\int_{B_j}\omega,\quad\iint\sigma(C_i)\wedge\omega=\int_{C_i}\omega$$

for every harmonic differential ω. These belong to Γ_{hic} and set

$$\theta_j=-{}^*\sigma(A_j)+\mathrm{i}\sigma(A_j),\ \ \theta_{n+j}=-{}^*\sigma(B_j)+\mathrm{i}\sigma(B_j),\ \ \theta_{2n+i}=-{}^*\sigma(C_i)+\mathrm{i}\sigma(C_i).$$

Theorem 4.[9] *Assume that every* $\log\|\theta^t_j\|$ *is harmonic with respect to* t. *Then* S *and* S_t *are the same Riemann surfaces.*

Proof. Using this assumption it follows that $\|c_i\theta^t_i+c_j\theta^t_j\|$ are constant. Therefore

(θ_i^t, θ_j^t) are constant and $\int_{f_t(A_j)}\theta_i^t$, $\int_{f_t(B_j)}\theta_i^t$ and $\int_{f_t(C_j)}\theta_i^t$ are constant. Applying Torelli's theorem to the double $\tilde{S}_t$ of S_t, we can obtain the result.

5. Torelli's theorem on Abelian Teichmüller disks

Let $\{A_j^t, B_j^t\}$ be a canonical homology basis of S_t of modulo dividing cycles, which A_j^t and B_j^t are homologous to $f_t(A_j), f_t(B_j)$ respectively ($S_0=S$, $A_j^0=A_j$, $B_j^0=B_j$). We can take a subspace $\Gamma_x(S)$[4] of $\Gamma_h(S)$ such that $\Gamma_x(S)={}^*\Gamma_x(S)^{\perp}$ and $\omega\in\Gamma_x(S)$ has vanishing A periods and semiexact, i.e.

$$\int_{A_j}\omega=0 \text{ for every } A_j,$$

$$\int_{C_j}\omega=0 \text{ for every dividing cycle } C_j.$$

Let $\Gamma_x(S_t)$ be the orthogonal projection to $\Gamma_h(S_t)$ of the pull back $\Gamma_x(S)\circ f_t^{-1}$. Then $\Gamma_x(S_t)$ satisfies the same conditions as $\Gamma_x(S)$. A holomorphic differential ϕ on S_t is said to have Λ_x-behavior (normal behavior[4]) if there exist $\omega\in\Lambda_x(S_t)$ and $\omega_0\in\Lambda_{e0}(S_t)$ such that $\phi=\omega+\omega_0$ outside of a compact set on S_t. We showed that there uniquely exists a holomorphic differential ϕ_i^t with Λ_x-behavior such that

$$\int_{A_j^t}\phi_i^t=\delta_{ij}.$$

We know $\phi_i^t\circ f_t-\phi_i^0\in\Lambda_x(S)+\Lambda_{e0}(S)$. Set $\pi_{ij}(t\mu)=\int_{B_j^t}\phi_i^t$. We have $\pi_{ij}(t\mu)=\pi_{ji}(t\mu)$ and

$$\pi_{ij}(t\mu)-\pi_{ij}(0)=\langle\phi_i^t\circ f_t-\phi_i^0, {}^*\bar{\phi}_j^0\rangle-\mathrm{i}\langle\mathrm{i}(\phi_i^t\circ f_t-\phi_i^0), {}^*\bar{\phi}_j^0\rangle.$$

Hence by Theorem B

$$\frac{\partial}{\partial\bar{t}}\pi_{ij}(t\mu)=\frac{1}{2}\overline{\iint(\phi_i^t\phi_j^t+\mathrm{i}^2\phi_i^t\phi_j^t)\mu\zeta_z^2\mathrm{d}z\mathrm{d}\bar{z}}=0.$$

Thus $\pi_{ij}(t\mu)$ is a holomorphic function with respect to t and

$$\frac{\partial}{\partial t}\pi_{ij}(t\mu)=\iint\phi_i^t\phi_j^t\mu\zeta_z^2\mathrm{d}z\mathrm{d}\bar{z}.$$

Let sketch I. Kra's arguments[3] in our circumstances. Take period reproducing closed real differentials α_j, β_j such that for every closed differential ω

$$\int_{A_j}\omega=\iint\alpha_j\wedge\omega,\quad \int_{B_j}\omega=\iint\beta_j\wedge\omega.$$

Set $\alpha_j^t=\alpha_j\circ f_t^{-1}$, $\beta_j^t=\beta_j\circ f_t^{-1}$. Then for every closed differential ω^t on S_t

$$\int_{A_j^t}\omega^t=\iint\alpha_j^t\wedge\omega^t,\quad \int_{B_j^t}\omega^t=\iint\beta_j^t\wedge\omega^t.$$

Since α^t, β^t are associated to a canonical homology bais, they satisfy

$$\iint\alpha_i^t\wedge\beta_j^t=\delta_{ij},\quad \iint\alpha_i^t\wedge\alpha_j^t=\iint\beta_i^t\wedge\beta_j^t=0.$$

For a holomorphic differential ψ with Λ_x-behavior, put

$$\iint \alpha_j\wedge\psi=x_j+iy_j,\quad \iint \beta_j\wedge\psi=\xi_j+i\eta_j\quad (x_j,\ y_j,\ \xi_j \text{ and } \eta_j \text{ are real}).$$

If $y_j=0$, set $\alpha_j(\psi)=\alpha_j$. $\beta_j(\psi)=\beta_j$. If $y_j\neq 0$, set $\alpha_j(\psi)=-\alpha_j\eta_j/y_j+\beta_j$, $\beta_j(\psi)=-\alpha_j$. Then $\iint \alpha_j(\psi)\wedge\psi=a_j$ are real and only finite number of them do not vanish. Further $\{\alpha_j^t(\psi)=\alpha_j(\psi)\circ f_t^{-1},\ \beta^t(\psi)=\beta_j(\psi)\circ f_t^{-1}\}$ becomes a canonical cohomology basis which satisfy

$$\iint \alpha_i^t(\psi)\wedge\beta_j(\psi)=\delta_{ij},\quad \iint \alpha_i^t(\psi)\wedge\alpha_j^t(\psi)=\iint \beta_i^t(\psi)\wedge\beta_j^t(\psi)=0.$$

We can take a holomorphic differential $\phi_i^t(\psi)$ with Λ_x-behavior such that

$$\iint \alpha_j^t(\psi)\wedge\phi_i^t(\psi)=\delta_{ij}.$$

Set $\pi_{ij}(t\mu,\psi)=\iint \beta_j^t(\psi)\wedge\phi_i^t(\psi)$. Now to simplify notation we set

$$\alpha^t=\begin{bmatrix}\alpha_1^t\\ \cdots\\ \alpha_n^t\end{bmatrix},\ \beta^t=\begin{bmatrix}\beta_1^t\\ \cdots\\ \beta_n^t\end{bmatrix},\ \alpha^t(\psi)=\begin{bmatrix}\alpha_1^t(\psi)\\ \cdots\\ \alpha_n^t(\psi)\end{bmatrix},\ \beta^t(\psi)=\begin{bmatrix}\beta_1^t(\psi)\\ \cdots\\ \beta_n^t(\psi)\end{bmatrix},$$

and represent the relation by matrix

$$\begin{bmatrix}\alpha^t(\psi)\\ \beta^t(\psi)\end{bmatrix}=\begin{bmatrix}K(\psi) & L(\psi)\\ M(\psi) & N(\psi)\end{bmatrix}\begin{bmatrix}\alpha^t\\ \beta^t\end{bmatrix},$$

where $K(\psi)$, $L(\psi)$, $M(\psi)$, $N(\psi)$ are $(n,\ n)$-matrices.

Since α^t, β^t and $\alpha^t(\psi)$, $\beta^t(\psi)$ are canonical cohomology basis,

$$\begin{bmatrix}K(\psi) & L(\psi)\\ M(\psi) & N(\psi)\end{bmatrix}\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix}{}^tK(\psi) & {}^tM(\psi)\\ {}^tL(\psi) & {}^tN(\psi)\end{bmatrix}=\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix},$$

where I is the unit matrix of degree n and ${}^t\!\sim$ denotes the transposed matrix. We have

$$\iint \alpha_i^t(\psi)\wedge\phi_j^t=k_{ij}(\psi)+\sum l_{ik}(\psi)\pi_{kj}(t\mu),\ (K(\psi)=[k_{ij}(\psi)],\ L(\psi)=[l_{ij}(\psi)]),$$

$$\iint \beta_i^t(\psi)\wedge\phi_j^t=m_{ij}(\psi)+\sum n_{ik}(\psi)\pi_{kj}(t\mu),\ (M(\psi)=[m_{ij}(\psi)],\ N(\psi)=[n_{ij}(\psi)]).$$

Since $\alpha_i^t(\psi)$ (resp. ϕ_j^t) are linearly independent, matrix $\left[\iint \alpha_i^t(\psi)\wedge\phi_j^t\right]$ is regular. Thus

$$(\phi_1^t(\psi)\ldots\phi_n^t(\psi))=(\phi_1^t\ldots\phi_n^t)[K(\psi)+L(\psi)\Pi(t\mu)]^{-1},\ (\Pi(t\mu)=[\pi_{ij}(t\mu)]),$$
$$\Pi(t\mu,\psi)=[M(\psi)+N(\psi)\Pi(t\mu)][K(\psi)+L(\psi)\Pi(t\mu)]^{-1},\ (\Pi(t\mu,\psi)=[\pi_{ij}(t\mu,\psi)]).$$

Put $\psi^t=\sum a_j\phi_j^t(\psi)$ and $\iint \beta_i^t(\psi)\wedge\psi^t=b_i(t,\psi)$. Consider

$$g(t\mu,\ \psi)=\sum a_ib_i(t,\ \psi)=\sum a_ia_j\pi_{ij}(t\mu,\ \psi).$$

Note that if $\omega,\ \sigma\in\Lambda_x+\Lambda_{eo}$, we have

$$0=(\omega,\ {}^*\sigma)=-\lim_{n\to\infty}\left\{\int_{\partial G_n} w\bar{\sigma}+\sum\left[\int_{A_j}\omega\int_{B_j}\bar{\sigma}-\int_{B_j}\omega\int_{A_j}\bar{\sigma}\right]\right\}$$

$$=-\lim_{n\to\infty}\int_{\partial G_n} w\bar{\sigma},$$

where $\omega=dw$ and $\{G_n\}$ is a canonical regular exhaustion associated to $\{A_j, B_j\}$. We know that Riemann's bilinear relation is valid for holomorphic differential with Λ_x-behavior. Therefore

$$\begin{aligned}\|\psi^t\|^2&=i\sum\left[\int_{A_j^t}\psi^t\int_{B_j^t}\bar{\psi}^t-\int_{B_j^t}\psi^t\int_{A_j^t}\bar{\psi}^t\right]\\&=i\sum\left[\iint a_j^t(\psi)\wedge\psi^t\iint\beta_j^t(\psi)\wedge\bar{\psi}^t-\iint\beta_j^t(\psi)\wedge\psi^t\iint a_j^t(\psi)\wedge\bar{\psi}^t\right]\\&=i\sum\left[a_j\overline{b_j(t,\psi)}-b_j(t,\psi)a_j\right]\\&=2\operatorname{Im} g(t\mu,\psi).\end{aligned}$$

Hence $g(t\mu, \psi)$ has a positive imaginary part. Next $\omega=\phi_i^t(\psi)\circ f_t-\phi_i(\psi)$ and $\sigma=\phi_j(\psi)-\beta_j(\psi)$ coincide with element in $\Lambda_x+\Lambda_{e0}$ outside of a compact set. We have

$$\begin{aligned}(\omega, {}^*\bar{\sigma})&=-\sum\left[\int_{A_k}\omega\int_{B_k}\sigma-\int_{B_k}\omega\int_{A_k}\sigma\right]\\&=-\sum\left[\iint a_k(\psi)\wedge\omega\iint\beta_k(\psi)\wedge\sigma-\iint\beta_k(\psi)\wedge\omega\iint a_k(\psi)\wedge\sigma\right]=0.\end{aligned}$$

It follows that

$$\begin{aligned}\pi_{ij}(t\mu,\psi)-\pi_{ij}(0,\psi)&=(\phi_i^t(\psi)\circ f_t-\phi_i(\psi), {}^*\beta_j(\psi))\\&=(\phi_i^t(\psi)\circ f_t-\phi_i(\psi), {}^*\bar{\phi}_j(\psi)),\end{aligned}$$

and by Theorem B

$$\frac{\partial}{\partial t}\pi_{ij}(t\mu,\psi)=\iint \underline{\phi_i^t(\psi)}\underline{\phi_j^t(\psi)}\underline{\mu}\zeta_z^2 dz d\bar{z}.$$

Thus

$$g'(0,\psi)=\iint\psi\wedge\mu\psi.$$

Note that

$$\left|\iint\psi\wedge\mu\psi\right|\leqq\iint|\psi\wedge\bar{\psi}|,$$

and the equality occurs if and only if $|\mu|=1$ a.e. and $\mu\psi^2$ has a constant argument. If $\mu=\bar{\psi}/\psi$, then $g'(0,\psi)=-2\,i\operatorname{Im} g(0,\psi)$. By Schwarz's lemma g is a conformal mapping from the unit disk to the upper half plane.

Theorem 5. *Let ψ and ψ' be non zero holomorphic differentials with Λ_x-behavior. Assume that for complex numbers t and s in the open unit disk,*

$$\pi_{ij}(t\bar{\psi}/\psi)=\pi_{ij}(s\bar{\psi}'/\psi') \text{ for every } i \text{ and } j.$$

Then $t\bar{\psi}/\psi=s\bar{\psi}'/\psi'$.

Proof. From the assumption we get $\pi_{ij}(t\bar{\psi}/\psi,\psi)=\pi_{ij}(s\bar{\psi}'/\psi',\psi)$. Hence $g(t\bar{\psi}/\psi,\psi)=g(s\bar{\psi}'/\psi',\psi)$. If ψ' is not a constant multiple of ψ, by Schwarz's lemma $|t|<|s|$. Applying this to $g(\ ,\psi')$, we also get $|s|<|t|$. This is a contradiction. It follows the result.

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