Quasiconformal Variations of Green's and Neumann's Functions for Arbitrary Hyperbolic Riemann Surfaces

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Abstract

The purpose of this paper is to show the variational formulas of potential theoretic quantities on arbitrary Riemann surfaces under quasiconformal deformations. We have already given the variational formulas for Riemann's period matrices and certain abelian differentials. There we used a notion of behavior spaces in the Hilbert space over the complex number field. In this paper we shall use behavior spaces (Shiba's type) in the Hilbert space of first order complex differentials over the real number field. Applying the previous consideration to them, we can get a variational formula of Green's functions on arbitrary hyperbolic Riemann surfaces, which is an extension of the one by Guerrero on finite Riemann surfaces. Further we shall obtain the variational formulas for period reproducing differentials, slits mappings, Neumann's functions, Robin's constants and Bergmann kernels.

1. Introduction

The variational formula for Green's function has been investigated firstly by Hadamard and later by Schiffer and many other authors[12]. Recently Guerrero[4] discussed the variational formula on finite Riemann surfaces by using the quasiconformal mappings and Fuchsian groups, and he asked its generalization to arbitrary hyperbolic Riemann surfaces.

In this paper we shall study the variational formulas of potential theoretic quantities on arbitrary hyperbolic Riemann surfaces induced by quasiconformal deformations depending on a complex parameter. For such a general case, we have already given several variational formulas for the abelian differentials by using a certain (complex) behavior space[6],[8],[9]. Similar formulas can be obtained for Green's functions and other fundamental functions. However, in treating the real functions, it is appropriate to use the real behavior space. In the present paper we shall therefore use the (real) behavior space of Shiba's type and show the variational formulas for Green's and Neumann's functions, Robin's constants and Bergmann kernels.

2. Behavior spaces and quasiconformal mappings

2.1. Let R be a Riemann surface and $\tilde{\Lambda}=\tilde{\Lambda}(R)$ be the Hilbert space of square integrable complex differentials on R with the usual inner product defined by

$$(\lambda_1, \lambda_2)=\iint_R \lambda_1\wedge\bar{\lambda}_2{}^*=i\iint_R (a_1\bar{a}_2+b_1\bar{b}_2)dzd\bar{z},$$

where $\lambda_i=a_i(z)dz+b_i(z)d\bar{z}$ i=1, 2, for a local parameter z. We denote by $\bar{\lambda}$ the complex conjugate of λ and by λ^* the conjugate differential of λ. As Shiba did, we regard $\tilde{\Lambda}$ as a Hilbert space $\Lambda=\Lambda(R)$ over the real number field with a new inner product

$$\langle\lambda_1, \lambda_2\rangle=\text{Real part of } (\lambda_1, \lambda_2)$$ [13].

Let $\Gamma=\Gamma(R)$ be the subspace of Λ which consists of square integrable *real* differentials. As for the notations of subspaces we follow Ahlfors-Sario[3] and Shiba[13], for example, Λ_c, Λ_h, Λ_{hse}, Λ_{he} (Γ_c, Γ_h, Γ_{hse}, Γ_{he}) the subspaces of closed, harmonic, harmonic semiexact, harmonic exact differentials and Λ_{hm}, Λ_{ho}, Λ_{eo} (Γ_{hm}, Γ_{ho}, Γ_{eo}) the orthogonal complement of Λ^*_{hse}, Λ^*_{he} in Λ_h (Γ^*_{hse}, Γ^*_{he} in Γ_h) and of Λ_h in Λ_c (Γ_h in Γ_c). We know

Lemma **1.** (*cf.* *13*))

$$\Lambda_h=\Lambda_c\cap\Lambda_c{}^*,\ \Lambda=\Lambda_h+\Lambda_{eo}+\Lambda_{eo}{}^*,\ \Lambda_c=\Lambda_h+\Lambda_{eo}.$$

Let Γ_x be a subspace of Γ_h and $\Gamma_x{}^\perp$ be the orthogonal complement of Γ_x in Γ_h. We set $\Lambda_x=\Gamma_x+i\Gamma_x{}^{\perp *}$. Then Λ_x is a subspace of Λ_h. Now let ω_1, $\omega_2\in\Gamma_x$ and ω_3, $\omega_4\in\Gamma_x{}^\perp$. We have

$$\begin{aligned}&\langle\omega_1+i\omega_3{}^*,\ i(\omega_2+i\omega_4{}^*)^*\rangle\\&=\mathrm{Re}\{(\omega_1, \omega_4)+(\omega_3{}^*, \omega_2{}^*)-i[(\omega_1, \omega_2{}^*)-(\omega_3{}^*, \omega_4)]\}\\&=0.\end{aligned}$$

Thus we have an orthogonal decomposition of Λ_h.

$$\Lambda_h=\Lambda_x+i\Lambda_x{}^*.$$

Definition. A meromorphic differential ψ *on* R *is said to have a* Λ_x*-behavior if there exist a compact set* K *and differentials* $\lambda_1\in\Lambda_x$, $\lambda_0\in\Lambda_{eo}$ *such that* $\psi=\lambda_1+\lambda_0$ *on* $R-K$.

2.2. Let f be a quasiconformal homeomorphism from a Riemann surface R' to R. Then f induces an isomorphism $f^\sharp$ from $\Lambda(R)$ to $\Lambda(R')$:

$$f^\sharp(\lambda)=[a(z)z_\zeta+b(z)\bar{z}_\zeta]d\zeta+[a(z)z_{\bar{\zeta}}+b(z)\bar{z}_{\bar{\zeta}}]d\bar{\zeta}$$

where $\lambda=a(z)dz+b(z)d\bar{z}\in\Lambda(R)$, z, $\zeta(z\underset{f}{\longleftrightarrow}\zeta)$ are the local parameters on R, R' and z_ζ, $\bar{z}_\zeta$, $z_{\bar{\zeta}}$, $\bar{z}_{\bar{\zeta}}$ are distributional derivatives[6),11)]. Let P be a natural projection from Λ to Λ_h and set $f_h{}^\sharp=P\circ f^\sharp$. We shall make use of the following lemmas.

Lemma **2.** (*cf.* *10*), *11*)) *The mappings* $(f^{-1})^\sharp\circ f^\sharp$ *and* $(f^{-1})_h{}^\sharp\circ f_h{}^\sharp$ *are identity mappings on* Λ *and* Λ_h *respectively.*

Lemma **3**. (*cf.* 6))

$$(f^{\#}(\tau_1)^*, f^{\#}(\tau_2^*))_{R'} = (\tau_1, \tau_2)_R \quad \textit{for any } \tau_1, \tau_2 \in \Lambda(R),$$
$$(f_h^{\#}(\omega_1)^*, f_h^{\#}(\omega_2^*))_{R'} = (\omega_1, \omega_2)_R \quad \textit{for any } \omega_1, \omega_2 \in \Lambda_h(R).$$

Lemma **4**. (*cf.* 10), 11)) *The $f_h^{\#}$ preserves the subspaces Γ_h, Γ_{hse}, Γ_{he}, Γ_{ho} and Γ_{hm}.*
We set

$$\Lambda_{x^0} = \Gamma_{he} + i\Gamma_{ho}, \; \Lambda_{x^1} = \{0\} + i\Gamma_h, \; \Lambda_{x^2} = \Gamma_{hm} + i\Gamma_{hse}.$$

Then we have by Lemma 4,

Lemma **5**. $f_h^{\#}(\Lambda_{x^i}(R)) = \Lambda_{x^i}(R')$ $(i = 0, 1, 2)$.

3. Meromorphic differentials with a Λ_x-behavior

3.1. In order to get the variational formulas we need

Proposition **1**. (*cf.* 1), 7)) *Let f be a quasiconformal homeomorphism from R' to R and the Beltrami coefficient μ of f^{-1} satisfy $|\mu| \le k < 1$. Let F be a closed region in R which contains the support of μ. Assume that $f_h^{\#}(\Lambda_x(R)) = \Lambda_x(R')$. Let ψ (resp. ψ') have a Λ_x-behavior on R (resp. R'). Assume that ψ has no singularity on F and*

$$(f^{-1})^{\#}(\psi') - \psi \in \Lambda_x + \Lambda_{eo}.$$

Then we have

$$\text{(i)} \quad \|\psi'_\zeta \zeta_z dz - \psi\|_R = \|\Psi'_\zeta \mu \zeta_z d\bar{z}\|_R,$$

$$\text{(ii)} \quad \|\Psi'_\zeta \zeta_z dz\|_F \le \frac{1}{1-k} \|\psi\|_F,$$

$$\text{(iii)} \quad \|\Psi'_\zeta \zeta_z dz - \psi\|_F \le \frac{1}{1-k} \|\psi\|_F,$$

$$\text{(iv)} \quad \|(f^{-1})^{\#}(\psi') - \psi\|_F \le \frac{\sqrt{2}\,k}{1-k} \|\psi\|_F,$$

where Ψ (resp. Ψ') denotes a primitive function of ψ on $R - \{A_j, B_j\}$ (resp. ψ' on $R' - \{A_j', B_j'\}$), ($\{A_j, B_j\}$ is a canonical homology basis on R (mod ∂R) and so $\{A_j', B_j'\}$, the image of $\{A_j, B_j\}$ by f forms a canonical homology basis on R' (mod $\partial R'$)).

Proof. Since $\Lambda_x + \Lambda_{eo}$ is orthogonal to $i\Lambda_x^* + \Lambda_{eo}^*$, we have

$$0 = \langle (f^{-1})^{\#}(\psi') - \psi, \; i((f^{-1})^{\#}(\psi') - \psi)^* \rangle$$
$$= \mathrm{Re}\, i \iint_R (|\Psi'_\zeta \zeta_z - \Psi_z|^2 - |\Psi'_\zeta \zeta_{\bar{z}}|^2) dz d\bar{z}.$$

Thus

$$\|\Psi'_\zeta \zeta_z dz - \psi\|_R = \|\Psi'_\zeta \mu \zeta_z d\bar{z}\|_R$$
$$\le k \|\Psi'_\zeta \zeta_z dz\|_F.$$

Therefore

$$\|\Psi'_\zeta \zeta_z dz\|_F - \|\psi\|_F \le k \|\Psi'_\zeta \zeta_z dz\|_F,$$

and

$$\|\Psi'_\zeta \zeta_z dz\|_F \leq \frac{1}{1-k}\|\psi\|_F.$$

Hence

$$\|\Psi'_\zeta \zeta_z dz - \psi\|_R \leq \frac{k}{1-k}\|\psi\|_F.$$

We have

$$\begin{aligned}
&\|(f^{-1})^{\sharp}(\psi')-\psi\|_R^2 \\
&= i\iint_R (|\Psi'_\zeta \zeta_z - \Psi_z|^2 + |\Psi'_\zeta \zeta_{\bar z}|^2) dz d\bar z \\
&= 2\|\Psi'_\zeta \mu \zeta_z d\bar z\|_R^2 \\
&\leq \frac{2k^2}{(1-k)^2}\|\psi\|_F^2.
\end{aligned}$$

This completes the proof.

3.2. Let $p \in R$ and V_1 be a parametric disc about p with local variable z. We set $V_r = \{p' \in V_1;\ |z(p')| < r\}\ (0 < r \leq 1)$ and $P_n = \{p, q\}$ for n=0, $\{p\}$ for n≥1. Take a $q \in V_{1/2}$. Then there exist functions $s_n \in C^2(R-P_n)$ such that

$$s_0 = \begin{cases} \log\left|\dfrac{z}{z-z(q)}\right| & \text{on } \bar V_{1/2} \\ 0 & \text{on } R-V_1, \end{cases}$$

$$s_n = \begin{cases} -\dfrac{1}{n}\operatorname{Re}\dfrac{1}{z^n} & \text{on } \bar V_{1/2} \\ 0 & \text{on } R-V_1 \end{cases} \qquad (n \geq 1).$$

Denote $ds_n = \sigma_n$. Now since $\int_{|z|=1/2} \sigma_n{}^* = \int_{|z|=1} \sigma_n{}^* = 0$, there exists a C^1-closed differential $\tilde\sigma_n$ such that $\tilde\sigma_n = \sigma_n{}^*$ on $(R-V_1) \cup \bar V_{1/2}$. Then $\sigma_n + \tilde\sigma_n{}^* \in \Lambda$ and $\sigma_n + \tilde\sigma_n{}^* = 0$ on $(R-V_1) \cup \bar V_{1/2}$. By the orthogonal decomposition we can write

$$\sigma_n + \tilde\sigma_n{}^* = \lambda_n + \tilde\lambda_n,\ \lambda_n \in \Lambda_x + \Lambda_{eo},\ \tilde\lambda_n \in i\Lambda_x{}^* + \Lambda_{eo}{}^*.$$

Set $\phi_n = \sigma_n - \lambda_n = \tilde\lambda_n - \tilde\sigma_n{}^*$. Then ϕ_n is closed and coclosed, hence ϕ_n is harmonic in $R-P_n$. Since $i\phi_n{}^* = i\tilde\lambda_n{}^*$ on $R-V_1$, the meromorphic differential $\psi_n = \phi_n + i\phi_n{}^*$ has a Λ_x-behavior. The ψ_0 (resp. ψ_n, n≥1) has singularities $\dfrac{dz}{z} - \dfrac{dz}{z-z(q)}$ $\left(\text{resp. } \dfrac{dz}{z^{n+1}}\right)$. Further note that

$$\psi_n - (\sigma_n + i\tilde\sigma_n) = \phi_n - \sigma_n + i(\phi_n{}^* - \tilde\sigma_n) = -\lambda_n + i\tilde\lambda_n{}^*,$$

and $\psi_n - (\sigma_n + i\tilde\sigma_n)$ belongs to $\Lambda_x + \Lambda_{eo}$. Assume that $\bar V_1$ does not meet F. Then we have

$$\begin{aligned}
&\langle (f^{-1})^{\sharp}(\psi') - \psi,\ i\overline{\psi_n}{}^* \rangle_{R-V_1} \\
&= \langle (f^{-1})^{\sharp}(\psi') - \psi,\ i(\overline{\psi_n - (\sigma_n + i\tilde\sigma_n)})^* \rangle_{R-V_1} \\
&= -\langle (f^{-1})^{\sharp}(\psi') - \psi,\ i(\overline{\psi_n - (\sigma_n + i\tilde\sigma_n)})^* \rangle_{V_1}
\end{aligned}$$

$$=-\mathrm{Re}\, i\int_{\partial V_1}(\Psi'\circ f^{-1}-\Psi)\psi_n$$

$$=\begin{cases}2\pi\,\mathrm{Re}\{\Psi'\circ f^{-1}(p)-\Psi'\circ f^{-1}(q)-(\Psi(p)-\Psi(q)\} & \text{for } n=0\\ \frac{2\pi}{n!}\,\mathrm{Re}\left\{\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi'\circ f^{-1}(p)-\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi(p)\right\} & \text{for } n\geq 1.\end{cases}$$

Now let $\tilde{V}_\varepsilon=\{p';\ |z(p')|<\varepsilon\}\cup\{p';\ |z(p')-z(q)|<\varepsilon\}$. Then

$$-\lim_{\varepsilon\to 0} i\int_{\partial\tilde{V}_\varepsilon}(\Psi'\circ f^{-1}-\Psi)\psi_0$$
$$=2\pi\{(\Psi'\circ f^{-1}(p)-\Psi(p))-(\Psi'\circ f^{-1}(q)-\Psi(q))\}.$$

Hence, even if F contains p and q, we have

$$\lim_{\varepsilon\to 0}\langle (f^{-1})^\sharp(\psi')-\psi,\ i\overline{\psi_0{}^*}\rangle_{R-\tilde{V}_\varepsilon}$$
$$=2\pi\,\mathrm{Re}\{\Psi'\circ f^{-1}(p)-\Psi(p)-(\Psi'\circ f^{-1}(q)-\Psi(q))\}.$$

Hereafter, the singular integral $(\omega,\sigma)_R$ means the Cauchy's principal value $\lim_{\varepsilon\to 0}(\omega,\sigma)_{R-\tilde{V}_\varepsilon}$ if it has a finite value.

Proposition 2. *Let* $(f^{-1})^\sharp(\psi')-\psi\in\Lambda_x+\Lambda_{eo}$. *Then*

$$\langle (f^{-1})^\sharp(\psi')-\psi,\ i\overline{\psi_0}{}^*\rangle_R$$
$$=2\pi\,\mathrm{Re}\{\Psi'\circ f^{-1}(p)-\Psi(p)-(\Psi'\circ f^{-1}(q)-\Psi(q))\}.$$

If the support of μ *does not meet* $V_\varepsilon=\{p';\ |z(p')|<\varepsilon\}$,

$$\langle (f^{-1})^\sharp(\psi')-\psi,\ i\overline{\psi_n{}^*}\rangle_R$$
$$=\frac{2\pi}{n!}\,\mathrm{Re}\left\{\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi'\circ f^{-1}(p)-\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi(p)\right\}\ \text{for } n\geq 1.$$

4. The variational formula for for the real part of a Λ_x-behavior

Let $\mu(z,t)\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}$ be a Beltrami differential with a complex parameter t on R and t varies a neighbourhood of zero in the complex plane. Assume that $\mu(z,t)$ ($\mu(z,0)=0$, $\|\mu(z,t)\|_\infty<1$) is analytic with respect to t for a fixed z and $\frac{\partial}{\partial t}\mu(z,t)$ is bounded and measurable. Now let R_t be a Riemann surface and f_t be a quasiconformal homeomorphism from R_t to R such that

$$\frac{(f_t{}^{-1})_{\bar{z}}}{(f_t{}^{-1})_z}=\frac{\zeta_{\bar{z}}}{\zeta_z}=\mu(z,t).$$

Assume that $(f_t)_h{}^\sharp(\Lambda_x(R))=\Lambda_x(R_t)$ and there exists a meromorphic differential ψ^t on R_t with a Λ_x-behavior such that

$$(f_t{}^{-1})^\sharp(\psi^t)-\psi\in\Lambda_x(R)+\Lambda_{eo}(R).$$

We also denote by Ψ^t the primitive function of ψ^t as mentioned before and by $\Psi_n{}^t$ that of $\psi_n{}^t=(\psi_n)^t$. For simplicity we omit t in these notations when $t=0$. Here we have the following variational formulas.

Lemma **6.** (*cf.* 5)) *Let* $\|\mu(z, t)\|_\infty=\text{esssup}\ |\mu(z, t)|=k(t)\leq k<1.$ *If* $\overline{\lim_{t\to 0}}\frac{k(t)}{|t|}<\infty$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}t}((f_t^{-1})^\sharp(\psi^t)-\psi,\ i\overline{\psi_n^*})_{R-\tilde{V}_\varepsilon}|_{t=0}$$
$$=-i\iint_{R-\tilde{V}_\varepsilon}(\Psi)_z(\Psi_n)_z\frac{\partial}{\partial t}\mu(z, 0)\mathrm{d}z\mathrm{d}\bar{z}\qquad (n=0, 1, 2, \ldots).$$

Proof. We first remark that

$$((f_t^{-1})^\sharp(\psi^t)-\psi,\ i\overline{\psi_n^*})_{R-\tilde{V}_\varepsilon}$$
$$=\iint_{R-\tilde{V}_\varepsilon}\{((\Psi^t)_\zeta\zeta_z-\Psi_z)\mathrm{d}z+(\Psi^t)_\zeta\zeta_{\bar{z}}\mathrm{d}\bar{z}\}\wedge i(\Psi_n)_z\mathrm{d}z$$
$$=-i\iint_{R-\tilde{V}_\varepsilon}(\Psi_n)_z(\Psi^t)_\zeta\zeta_{\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}$$
$$=-i\iint_{F-\tilde{V}_\varepsilon}(\Psi_n)_z\Psi_z\mu(z, t)\mathrm{d}z\mathrm{d}\bar{z}$$
$$-i\iint_{F-\tilde{V}_\varepsilon}(\Psi_n)_z((\Psi^t)_\zeta\zeta_z-\Psi_z)\mu(z, t)\mathrm{d}z\mathrm{d}\bar{z}.$$

Next we have

$$\lim_{t\to 0}\left|-i\iint_{F-\tilde{V}_\varepsilon}(\Psi_n)_z((\Psi^t)_\zeta\zeta_z-\Psi_z)\frac{\mu(z, t)}{t}\mathrm{d}z\mathrm{d}\bar{z}\right|$$
$$\leq\lim_{t\to 0}\frac{k(t)}{|t|}\left|-i\iint_{F-\tilde{V}_\varepsilon}(\Psi_n)_z((\Psi^t)_\zeta\zeta_z-\Psi_z)\mathrm{d}z\mathrm{d}\bar{z}\right|$$
$$\leq\lim_{t\to 0}\frac{k(t)}{|t|}\|\psi_n\|_{F-\tilde{V}_\varepsilon}\|(\Psi^t)_\zeta\zeta_z\mathrm{d}z-\psi\|_{F-\tilde{V}_\varepsilon}$$
$$\leq\|\psi_n\|_{F-\tilde{V}_\varepsilon}\lim_{t\to 0}\frac{k(t)^2}{(1-k(t))|t|}\|\psi\|_F$$
$$=0.$$

Thus we can conclude the assertion.

Proposition **3.** *Let* $\|\mu(z. t)\|_\infty\leq k<1$ *and* $\lim_{\tau\to 0}\frac{1}{|\tau|}|\mu(z, t+\tau)-\mu(z, t)|<\infty.$ *Assume that the support of* μ *does not contain the points* p *and* q. *Let* ψ_n *and* ψ_n^t *have the same singularities at* p *and* q *such that* $(f_t^{-1})^\sharp(\psi_n^t)-\psi_n\in\Lambda_x+\Lambda_{eo}$. *Then*

$$\frac{\partial}{\partial t}\mathrm{Re}\{\Psi^t\circ f_t^{-1}(p)-\Psi^t\circ f_t^{-1}(q)\}$$
$$=\frac{-i}{4\pi}\iint_R(\Psi^t)_\zeta\circ f_t^{-1}(\Psi_0^t)_\zeta\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z, t)\zeta_z^2\mathrm{d}z\mathrm{d}\bar{z},$$
$$\frac{\partial}{\partial t}\mathrm{Re}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi^t\circ f_t^{-1}(p)$$
$$=-\frac{n!}{4\pi}i\iint_R(\Psi^t)_\zeta\circ f_t^{-1}(\Psi_n^t)_\zeta\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z, t)\zeta_z^2\mathrm{d}z\mathrm{d}\bar{z}.$$

Proof. Take a $\tilde{V}_\varepsilon$ which does not meet the support of μ. We have by Lemmas 1 and 2,

$$((f_{t+\tau}^{-1})^{\sharp}(\psi^{t+\tau})-(f_t^{-1})^{\sharp}(\psi^t),\ i\overline{\psi_n{}^*})_{R-\tilde{V}_\varepsilon}$$
$$=(f_t^{\sharp}\circ(f_{t+\tau}^{-1})^{\sharp}(\psi^{t+\tau})-\psi^t,\ i\overline{(f_t^{\sharp}(\psi_n))^*})_{R_t-\tilde{V}_\varepsilon{}^t}\quad(\tilde{V}_\varepsilon{}^t=f_t(\tilde{V}_\varepsilon)).$$

Since $f_t^{\sharp}\circ(f_{t+\tau}^{-1})^{\sharp}(\psi^{t+\tau})-\psi^t\in\Lambda_x(R_t)+\Lambda_{eo}(R_t)$ and $i(\overline{f_t^{\sharp}(\psi_n)-\psi_n{}^t})^*\in i(\Lambda_x(R_t)^*+\Lambda_{eo}(R_t)^*)$, we have

$$\langle(f_{t+\tau}^{-1})^{\sharp}(\psi^{t+\tau})-(f_t^{-1})^{\sharp}(\psi^t),\ i\overline{\psi_n{}^*}\rangle_{R-\tilde{V}_\varepsilon}$$
$$=\langle(f_{t+\tau}^{-1}\circ f_t)^{-1\sharp}(\psi^{t+\tau})-\psi^t,\ i\overline{\psi_n{}^{t*}}\rangle_{R_t-\tilde{V}_\varepsilon{}^t}$$

The Beltrami differential of $f_{t+\tau}^{-1}\circ f_t$ is the following[2)]

$$\frac{\zeta_z}{\overline{\zeta_z}}\frac{\mu(z,t+\tau)-\mu(z,t)}{1-\overline{\mu(z,t)}\mu(z,t+\tau)}\circ f_t\frac{d\bar{\zeta}}{d\zeta}.$$

If we denote this by $\nu(\zeta,\tau)\dfrac{d\bar{\zeta}}{d\zeta}$, by Lemma 6 we have for $\tau=u+iv$

$$\frac{\partial}{\partial u}\langle(f_{t+\tau}^{-1}\circ f_t)^{-1\sharp}(\psi^{t+\tau})-\psi^t,\ i\overline{\psi_n{}^{t*}}\rangle_{R_t-\tilde{V}_\varepsilon{}^t}$$
$$=\mathrm{Re}\frac{d}{d\tau}((f_{t+\tau}^{-1}\circ f_t)^{-1\sharp}(\psi^{t+\tau})-\psi^t,\ i\overline{\psi_n{}^{t*}})_{R_t-\tilde{V}_\varepsilon{}^t}$$
$$=-\mathrm{Re}\, i\iint_{R_t-\tilde{V}_\varepsilon{}^t}(\Psi^t)_\zeta(\Psi_n{}^t)_\zeta\lim_{\tau\to0}\frac{\nu(\zeta,\tau)}{\tau}d\zeta d\bar{\zeta}$$
$$=-\mathrm{Re}\, i\iint_{R_t-\tilde{V}_\varepsilon{}^t}(\Psi^t)_\zeta(\Psi_n{}^t)_\zeta\frac{\zeta_z}{\overline{\zeta_z}}\frac{\mu_t(z,t)}{1-|\mu(z,t)|^2}\circ f_t\,d\zeta d\bar{\zeta}$$
$$=-\mathrm{Re}\, i\iint_{R-\tilde{V}_\varepsilon}(\Psi^t)_\zeta\circ f_t^{-1}(\Psi_n{}^t)_\zeta\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,dzd\bar{z}.$$

Similarly we have

$$\frac{\partial}{\partial v}\langle(f_{t+\tau}^{-1}\circ f_t)^{-1\sharp}(\psi^{t+\tau})-\psi^t,\ i\overline{\psi_n{}^{t*}}\rangle_{R_t-\tilde{V}_\varepsilon{}^t}$$
$$=-\mathrm{Re}\, i\iint_{R-\tilde{V}_\varepsilon}(\Psi^t)_\zeta\circ f_t^{-1}(\Psi_n{}^t)_\zeta\circ f_t^{-1}\, i\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,dzd\bar{z}.$$

Hence

$$\frac{\partial}{\partial t}\langle(f_t^{-1})^{\sharp}(\psi^t)-\psi,\ -i\overline{\psi_n{}^*}\rangle_{R-\tilde{V}_\varepsilon}$$
$$=-\frac{i}{2}\iint_{R_t-\tilde{V}_\varepsilon{}^t}(\Psi^t)_\zeta(\Psi_n{}^t)_\zeta\frac{\zeta_z}{\overline{\zeta_z}}\frac{\mu_t(z,t)}{1-|\mu(z,t)|^2}\circ f_t\,d\zeta d\bar{\zeta}$$
$$=-\frac{i}{2}\iint_{R-\tilde{V}_\varepsilon}(\Psi^t)_\zeta\circ f_t^{-1}(\Psi_n{}^t)_\zeta\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,dzd\bar{z}.$$

From Proposition 2 the statement follows.

Let a cycle C in R be represented as $\sum_{i=0}^{n}[p_i,p_{i+1}]$ and set $\psi_c=\sum_{i=0}^{n}\psi_{p_i,p_{i+1}}$, where $\psi_{p_i,p_{i+1}}=\psi_0$ for $q=p_i$, $p=p_{i+1}$. Let denote by ψ_{Ct}^t the differential on R_t corresponding to ψ_C which has a Λ_x-behavior.

Corollary Let C not meet the singularities of ψ. Then

$$\frac{\partial}{\partial t}\,\mathrm{Re}\int_{C^t}\psi^t=\frac{-i}{4\pi}\iint_R(\Psi^t)_\zeta\circ f_t^{-1}(\Psi'^t_{C^t})_\zeta\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z}.$$

Remark. Since $(f_t^{-1})^\sharp(\psi^t_{C^t})-\psi_C\in\Lambda_x+\Lambda_{eo}$, this variational formula is valid for the case that the support of μ meets C.

5. The case of Green's and Neumann's functions

Some slit mappings on finite bordered Riemann surfaces are given by meromorphic differentials with a Λ_{x^2}-behavior[13] and the variational formulas are given in Proposition 3. Now we shall show that the variational formulas for Green's and Neumann's functions are those of meromorphic differentials with the Λ_{x^1} and the Λ_{x^0}-behavior respectively. Let $G^t_{p^t}$ (resp. $G^t_{q^t}$) be the Green's function on R_t with the pole at $p^t=f_t^{-1}(p)$ (resp. $q^t=f_t^{-1}(q)$) and $N^t_{p^t,q^t}$ be the Neumann's function on R_t with poles at p_t and q_t. Write $G^t_{p^t,q^t}=G^t_{p^t}-G^t_{q^t}$.

Lemma 7. *The differential* $\mathrm{d}(G^t_{p^t,q^t})+i\mathrm{d}(G^t_{p^t,q^t})^*$ *has the Λ_{x^1}-behavior and* $\mathrm{d}(N^t_{p^t,q^t})+i\mathrm{d}(N^t_{p^t,q^t})^*$ *has the Λ_{x^0}-behavior.*

Proof. Let $\sigma_0{}^t$ and $\tilde{\sigma}_0{}^t$ be the differentials on R_t corresponding to σ_0 and $\tilde{\sigma}_0$ on R (see sec. 3). Note that $2d(G^t_{p^t,q^t})+(\sigma_0{}^t+\overline{\sigma_0{}^t})$ belongs to Γ_{eo} and $2d(G^t_{p^t,q^t})-(\tilde{\sigma}_0{}^t+\overline{\tilde{\sigma}_0{}^t})^*$ belongs to $\Gamma_h+\Gamma_{eo}{}^*$ because it is coclosed in $R_t-\{p^t,q^t\}$. Hence $\mathrm{d}(G^t_{p^t,q^t})+i\mathrm{d}(G^t_{p^t,q^t})^*$ has the Λ_{x^1}-behavior. As for the Neumann's function it is clear that $2d(N^t_{p^t,q^t})+(\sigma_0{}^t+\overline{\sigma_0{}^t})$ belongs to $\Gamma_{he}+\Gamma_{eo}$ and $2d(N^t_{p^t,q^t})-(\tilde{\sigma}_0{}^t+\overline{\tilde{\sigma}_0{}^t})^*$ is coclosed in $R_t-\{p^t,q^t\}$. Further we know

$$\langle 2\mathrm{d}(N^t_{p^t,q^t})-(\tilde{\sigma}_0{}^t+\overline{\tilde{\sigma}_0{}^t})^*,\ \mathrm{d}h\rangle=0 \text{ for } \mathrm{d}h\in\Gamma_{he}.$$

Hence $2\mathrm{d}(N^t_{p^t,q^t})-(\tilde{\sigma}_0{}^t+\overline{\tilde{\sigma}_0{}^t})^*$ belongs to $\Gamma_{ho}{}^*+\Gamma_{eo}{}^*$. Thus $\mathrm{d}(N^t_{p^t,q^t})+i\mathrm{d}(N^t_{p^t,q^t})^*$ has the Λ_{x^0}-behavior.

If the support of μ does not contain p and q, clearly we have that $(f_t^{-1})^\sharp(\mathrm{d}(G^t_{p^t,q^t})+i\mathrm{d}(G^t_{p^t,q^t})^*)-(\mathrm{d}(G_{p,q})+i\mathrm{d}(G_{p,q})^*)$ belongs to $\Lambda_{x^1}+\Lambda_{eo}$, and that

$$(f_t^{-1})^\sharp(\mathrm{d}(N^t_{p^t,q^t})+i\mathrm{d}(N^t_{p^t,q^t})^*)-(\mathrm{d}(N_{p,q})+i\mathrm{d}(N_{p,q})^*)$$

belongs to $\Lambda_{x^0}+\Lambda_{eo}$. By Proposition 3, we have

Proposition 4. *Let the support of μ not contain the points a, b, p and q. Then*

$$\frac{\partial}{\partial t}\{G^t_{a^t,b^t}(p^t)-G^t_{a^t,b^t}(q^t)\}$$

$$=-\frac{i}{\pi}\iint_R\frac{\partial}{\partial\zeta}(G^t_{a^t,b^t})\circ f_t^{-1}\frac{\partial}{\partial\zeta}(G^t_{q^t,p^t})\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z},$$

$$\frac{\partial}{\partial t}\{N^t_{a^t,b^t}(p^t)-N^t_{a^t,b^t}(q^t)\}$$

$$=-\frac{i}{\pi}\iint_R \frac{\partial}{\partial\zeta}(N^t_{a^t,b^t})\circ f_t^{-1}\frac{\partial}{\partial\zeta}(N^t_{q^t,p^t})\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,dzd\bar{z}.$$

Corollary. Assume that there exists a sequence $\{q_n\}\subset R-F$ *such that* $\lim_{n\to\infty}\sup_{p'\in F} G_{q_n}(p')=0$. *Then*

$$\frac{\partial}{\partial t}G^t_{a^t}(p^t)=\frac{i}{\pi}\iint_R \frac{\partial}{\partial\zeta}G^t_{a^t}\circ f_t^{-1}\frac{\partial}{\partial\zeta}G^t_{p^t}\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,dzd\bar{z}.$$

Proof. For a fixed m set $b_m=q_m$. Now let n tend to infinity. Note that $\lim_{n\to\infty} G^t_{a^t,b^t}(q_n{}^t)=0$ and $\|d(G^t_{q^t_n,p^t}+G^t_{p^t})\|_{f_t^{-1}(F)}$ converges uniformly to 0 with respect to t. Successively let m tend to infinity. Then $\lim_{m\to\infty} G^t_{a^t,b^t_m}(p^t)=G^t_{a^t}(p^t)$ and the right side of the formula in Proposition 4 ($b=b_m$, $q=q_n$) converges uniformly to that in Corollary with respect to t. Hence the assertion follows.

Let $V=\{z;\ |z|<1\}$ be a parametric disc about a. Assume that $\bar{V}$ does not meet the support of μ. Set

$$\gamma^t(a^t)=\frac{1}{2\pi i}\int_{|z|=1} G^t_{a^t}\circ f_t^{-1}(z)\frac{dz}{z}.$$

We call $\gamma^t(a^t)$ the Robin's constant of R_t at a^t. We have a variational formula for the Robin's constant. Note that

$$G^t_{a^t}(\zeta)=\frac{1}{2\pi i}\int_{|z|=1} G^t_\zeta(f_t^{-1}(z))\frac{dz}{z}.$$

Corollary.

$$\frac{\partial}{\partial t}\gamma^t(a^t)=\frac{i}{\pi}\iint_R\left(\frac{\partial}{\partial\zeta}G^t_{a^t}\right)^2\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,dzd\bar{z}.$$

6. The variational formula for the imaginary part of a Λ_x-behavior.

Let ψ_x be a meromorphic differential with a Λ_x-behavior and $\tilde{\psi}_x$ denote a meromorphic differential with a Λ_x-behavior whose singularities are the same as those of $i\psi_x$. Note that $\tilde{\psi}_x$ is not always unique for ψ_x. Let $\Lambda_y=\Gamma_x{}^{\perp *}+i\Gamma_x(=i\Lambda_x)$. Then we have

Lemma 7. The $-i\tilde{\psi}_x$ *is a meromorphic differential with a* Λ_y*-behavior whose singularities are the same as those of* ψ_x.

Proposition 5. Let $\tilde{\psi}^t_{x,n}=i\psi^t_{y,n}$. *Under the similar conditions as in Proposition 3, we have*

$$\frac{\partial}{\partial t}\operatorname{Im}\{\Psi_x{}^t\circ f_t^{-1}(p)-\Psi_x{}^t\circ f_t^{-1}(q)\}$$
$$=\frac{i}{4\pi}\iint_R(\Psi_x{}^t)_\zeta\circ f_t^{-1}(\tilde{\Psi}^t_{x,0})_\zeta\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,dzd\bar{z},$$
$$\frac{\partial}{\partial t}\operatorname{Im}\frac{d^n}{dz^n}\Psi_x{}^t\circ f_t^{-1}(p)$$
$$=\frac{n!}{4\pi}i\iint_R(\Psi_x{}^t)_\zeta\circ f_t^{-1}(\tilde{\Psi}^t_{x,n})_\zeta\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,dzd\bar{z}.$$

Proof. By lemma 7 and Proposition 3, we have

$$\frac{\partial}{\partial t}\operatorname{Im}\{\Psi_x{}^t\circ f_t{}^{-1}(p)-\Psi_x{}^t\circ f_t{}^{-1}(q)\}$$
$$=-\frac{\partial}{\partial t}\operatorname{Re}\{\tilde{\Psi}_y{}^t\circ f_t{}^{-1}(p)-\tilde{\Psi}_y{}^t\circ f_t{}^{-1}(q)\}\qquad(\tilde{\psi}_y{}^t=i\psi_x{}^t)$$
$$=\frac{i}{4\pi}\iint_R(\tilde{\Psi}_y{}^t)_\zeta\circ f_t{}^{-1}(\Psi^t_{y,0})_\zeta\circ f_t{}^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}z$$
$$=\frac{i}{4\pi}\iint_R(i\Psi_x{}^t)_\zeta\circ f_t{}^{-1}(-i\tilde{\Psi}^t_{x,0})_\zeta\circ f_t{}^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z}$$
$$=\frac{i}{4\pi}\iint_R(\Psi_x{}^t)_\zeta\circ f_t{}^{-1}(\tilde{\Psi}^t_{x,0})_\zeta\circ f_t{}^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z}.$$

Similarly we have the second equation.

We have from Propositions 3 and 5,

Proposition **6**. *Under the similar conditions as in Proposition 3,*

$$\frac{\partial}{\partial t}\{\Psi_x{}^t\circ f_t{}^{-1}(p)-\Psi_x{}^t\circ f_t{}^{-1}(q)\}$$
$$=-\frac{i}{4\pi}\iint_R(\Psi_x{}^t)_\zeta\circ f_t{}^{-1}(\Psi^t_{x,0}-i\tilde{\Psi}^t_{x,0})_\zeta\circ f_t{}^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z},$$
$$\frac{\partial}{\partial t}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi_x{}^t\circ f_t{}^{-1}(p)$$
$$=-\frac{n!}{4\pi}i\iint_R(\Psi_x{}^t)_\zeta\circ f_t{}^{-1}(\Psi^t_{x,n}-i\tilde{\Psi}^t_{x,n})_\zeta\circ f_t{}^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z}.$$

In particular, if we remark that $K^t=\frac{1}{4\pi}(\Psi^t_{x^1,1}+i\tilde{\Psi}^t_{x^1,1})$ is a Bergmann kernel, we can get the variational formulas for Bergmann kernels.

Corollary.

$$\frac{\partial}{\partial t}\{K^t\circ f_t{}^{-1}(p)-K^t\circ f_t{}^{-1}(q)\}$$
$$=-\frac{i}{4\pi}\iint_R(K^t)_\zeta\circ f_t{}^{-1}(\Psi^t_{x^1,0}-i\tilde{\Psi}^t_{x^1,0})_\zeta\circ f_t{}^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z},$$
$$\frac{\partial}{\partial t}\frac{\mathrm{d}^n}{\mathrm{d}z^n}K^t\circ f_t{}^{-1}(p)$$
$$=-\frac{n!}{4\pi}i\iint_R(K^t)_\zeta\circ f_t{}^{-1}(\Psi^t_{x^1,n}-i\tilde{\Psi}^t_{x^1,n})_\zeta\circ f_t{}^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z},$$

Similarly we have

Proposition 7.

$$\frac{\partial}{\partial t}\{\Psi_x{}^t\circ f_t{}^{-1}(p)-\Psi_x{}^t\circ f_t{}^{-1}(q)\}$$
$$=\frac{i}{4\pi}\overline{\iint_R(\Psi_x{}^t)_\zeta\circ f_t{}^{-1}(\Psi^t_{x,0}+i\tilde{\Psi}^t_{x,0})_\zeta\circ f_t{}^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z{}^2\,\mathrm{d}z\mathrm{d}\bar{z}},$$
$$\frac{\partial}{\partial t}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi_x{}^t\circ f_t{}^{-1}(p)$$

$$=\frac{n!}{4\pi}i\iint_R \overline{(\Psi_x^t)_\zeta\circ f_t^{-1}(\Psi^t_{x,n}+i\tilde{\Psi}^t_{x,n})_\zeta\circ f_t^{-1}\frac{\partial}{\partial t}\mu(z,t)\zeta_z^2\,dz d\bar{z}}$$

If $\Gamma_x=\Gamma_x^{\perp *}$, $\Lambda_x=\Gamma_x+i\Gamma_x$. We have $\Lambda_x=\Lambda_y$ and $\tilde{\psi}_x=i\psi_x$. Hence the right hand sides in equalities of Proposition 7 are zero. This shows the following[6),9)].

Corollary. Let $\Lambda_x=i\Lambda_x$. *Then the quantities* $\Psi_x^t\circ f_t^{-1}(p)-\Psi_x^t\circ f_t^{-1}(q)$ *and* $\frac{d^n}{dz^n}\Psi_x^t\circ f_t^{-1}(p)$ *are holomorphic with respect to t.*

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