Behavior Spaces on Finite Bordered Riemann Surfaces

By

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Abstract

We introduced in our previous work behavior spaces for the purpose of a formulation of the Riemann-Roch theorem and Abel's theorem on open Riemann surfaces. The existence of behavior spaces on arbitrary Riemann surfaces was shown by using Zorn's Lemma. But behavior spaces are not very clear. So as to throw light on it we shall give concretely some behavior spaces on finite bordered Riemann surfaces. If we use the specific kind of behavior space, our formulation of the theories is regarded as an obedient generalization of classical theories on compact surfaces. We shall give an example and show that meromorphic functions which is stated in our Riemann-Roch theorem are not always trivial on a surface with infinite genus and a large boundary. We shall give meromorphic functions on finite bordered Riemann surfaces whose real and imaginary parts have a certain Γ_χ-behavior. This was unexpected on Riemann surfaces with large boundaries.

1. Introduction

The classical Riemann-Roch theorem and Abel's theorem on compact Riemann surfaces have been generalized to open Riemann surfaces. A type of similar generalization is formulated on complex vector spaces as those given by L. Ahlfors[2,3], B. Royden[11], B. Rodin[10], Y. Sainouchi[12] and O. Watanabe[15]. But, as was pointed out by R. Accola, it seems to be meaningful for Riemann surfaces of class O_{KD} only. The other type is considered on real vector spaces as those given by Y. Kusunoki[6] firstly, M. Yoshida[18], M. Shiba[13,14] and O. Watanabe[16] afterward. This is valid for general Riemann surfaces and the grounds might become complex vector spaces if the concerned surfaces have certain small boundaries. These generalizations are formulated by the method of behavior spaces on the real number field and are concerned with meromorphic functions with a Γ_χ-behavior introduced by M. Yoshida. It has seemed that a meromorphic function whose real and imaginary parts have a Γ_χ-behavior is trivial if the concerned surface is not of class O_{KD}. Recently, by the method of behavior spaces on the complex number field, we formulated these theories on complex vector spaces which is not always trivial for surfaces even with large boundaries.

In this paper we shall investigate some kinds of behavior spaces concretely on finite bordered Riemann surfaces, and applying them we shall discuss Weierstrass points and

Jacobi's inversion problem on finite bordered Riemann surfaces from our view point. By considering a specific kind of behavior space we know that our formulation is regarded as an obedient generalization of classical theories on compact Riemann surfaces. We shall show by examples that a meromorphic function which is given by our Riemann-Roch theorem is not trivial on a surface with infinite genus and a large boundary, and that meromorphic differentials with an infinite number of non-vanishing periods may appear in our Abel's theorem. We can give some Γ_X-behaviors on finite bordered Riemann surfaces such that meromorphic functions whose real and imaginary parts have a Γ_X-behavior are not always trivial on these surfaces with large boundaries.

2. Behavior spaces

2·1 *X-behavior*

Let R be a Riemann surface and $\Gamma=\Gamma(R)$ be the Hilbert space of square integrable complex differentials on R with the usual inner product defined by

$$(\omega_1, \omega_2)=\iint_R \omega_1\wedge\bar{\omega}_2{}^*=i\iint_R (a_1\bar{a}_2+b_1\bar{b}_2)dzd\bar{z},$$

where $\omega_i=a_i dz+b_i d\bar{z}$ for a local parameter z. We denote by $\bar{\omega}$ the complex conjugate of ω and by ω^* the conjugate differential of ω. We shall make use of some kind of subspaces of Γ. As for the notation of subspaces we follow Ahlfors-Sario[3], for example, Γ_h, Γ_{hse}, Γ_{eo} denote the spaces of harmonic, harmonic semiexact differentials and the space of differentials of the Dirichlet potential respectively. Let $\Xi=\{A_j, B_j\}$ be a canonical homology basis of R modulo dividing cycles and $\{G_n\}$ be a canonical regular exhaustion to which Ξ is associated. We will take up the following type of subspaces.

Definition. A subspace Γ_x of Γ_h *is called a behavior space if* Γ_x *satisfies the following*;

(i) $\Gamma_x\subset\Gamma_{hse}$,

(ii) $\Gamma_x=\bar{\Gamma}_x=\Gamma_x{}^{\perp *}$,

(iii) *there exists a system of real numbers* $\{a_j, b_j\}$ $(|a_j|+|b_j|\neq 0)$ *such that*

$$a_j\int_{A_j}\omega=b_j\int_{B_j}\omega \qquad \textit{for any } \omega\in\Gamma_x.$$

We denote

$$\bar{\Gamma}_x=\{\bar{\omega};\ \omega\in\Gamma_x\},$$

$$\Gamma_x{}^*=\{\omega^*;\ \omega\in\Gamma_x\},$$

$$\Gamma_x{}^{\perp}=\{\omega\in\Gamma_h;\ (\omega, \sigma)=0 \text{ for any } \sigma\in\Gamma_x\}.$$

Well known subspaces do not become behavior spaces, but we have shown in[8] that

Proposition 1. On an arbitrary Riemann surface there exists a behavior space.

Now let $\{V_i\}$ ($V_i=\{z_i;\ |z_i|<1\}$ be a family of parametric discs on R such that $\bar{V}_i\cap\bar{V}_j=\phi$ for $i\neq j$ and $\{V_i\}$ does not accumulate in R. We put $G=\bigcup_i V_i$. Assume that $G\cap\partial G_n=\phi$ for any n. For a behavior space we set $X=\Gamma_x+\Gamma_{eo}$.

Definition. A meromorphic differential ψ *has an X-behavior if there exist a* G_n *and* *an* ω *in* X *such that* $\psi=\omega$ *on* $R-G-G_n$. We saw in[8] that there exist the following meromorphic differentials with an X-behavior:

ψ_{A_j} ; ψ_{A_j} is a holomorphic differential with an X-behavior such that
$a_i\int_{A_i}\psi_{A_j}=b_i\int_{B_i}\psi_{A_j}-b_i\delta_{ij}$ ($\delta_{ii}=1$, $\delta_{ij}=0$ for $i\neq j$),

ψ_{B_j} ; ψ_{B_j} is a holomorphic differential with an X-behavior such that
$a_i\int_{A_i}\psi_{B_j}=b_i\int_{B_i}\psi_{B_j}-a_i\delta_{ij}$,

$\psi_{p,n}$; $\psi_{p,n}$ is a meromorphic differential with an X-behavior such that $\psi_{p,n}$ has the singularity $\frac{dz}{z^{n+1}}$ only at p and $a_i\int_{A_i}\psi_{p,n}=b_i\int_{B_i}\psi_{p,n}$ $(n\geq 1)$,

$\psi_{p,q}$; $\psi_{p,q}$ is a meromorphic differential with an X-behavior such that $\psi_{p,q}$ has residue 1 at p and -1 at q, regular analytic elsewhere and $a_i\int_{A_i}\psi_{p,q}=b_i\int_{B_i}\psi_{p,q}$.

These differentials will play a fundamental role in the Riemann-Roch theorem and Abel's theorem

2·2 *Riemann-Roch theorem*

Let δ be a finite or infinite divisor on R whose support is contained in G and whose restriction to each V_i is a finite divisor. Write as $\delta=\delta_p/\delta_q$, where $\delta_p=p_1^{\mu_1}p_2^{\mu_2}\cdots p_n^{\mu_n}\cdots$ and $\delta_q=q_1^{\nu_1}q_2^{\nu_2}\cdots q_m^{\nu_m}\cdots$ are disjoint integral divisors. We consider the following linear spaces over the complex number field C:

$\mathrm{S}(X;\, 1/\delta) = \{f$; (i) f is a single-valued meromorphic function on R, (ii) df has an X-behavior, (iii) the divisor of f is a multiple of $1/\delta$.$\}$

$\mathrm{M}(X;\, 1/\delta_p) = \{f$; (i) f is a (multi-valued) meromorphic function on R such that $a_i\int_{A_i}df= b_i\int_{B_i}df$, (ii) df has an X-behavior, (iii) the divisor of f is a multiple of $1/\delta_p$.$\}$

$\mathrm{D}(X;\, \delta) = \{\psi$; (i) ψ is a meromorphic differential with an X-behavior, (ii) the divisor of ψ is a multiple of δ and ψ has a finite number of poles.$\}$

$\mathrm{D}(X;\, 1/\delta_q) = \{\psi$; (i) ψ is a meromorphic differential with an X-behavior, (ii) the divisor of ψ is a multiple of $1/\delta_q$ and ψ has a finite number of poles.$\}$

Here, in the case that $\delta_q\neq 1$ we identify two elements f_1 and f_2 of $\mathrm{M}(X;\, 1/\delta_p)$ if and only if f_1-f_2 is a constant.

Lemma 1. Let $f\in\mathrm{M}(X;\, 1/\delta_p)$ *and* $\psi\in\mathrm{D}(X;\, 1/\delta_q)$. *Then*

$$\lim_{n\to\infty}\int_{\partial G_n}f\psi=0 \quad \textit{and} \lim_{n\to\infty}\int_{\partial G_n}f\bar{\psi}=0.$$

Proof. There exist a region G_m and differentials ω and σ in X such that $df=\omega$ and $\psi=\sigma$ on $R-G_m-G$. Then we have

$$\begin{aligned}0&=(\omega,\, \bar{\sigma}^*)=\lim_{n\to\infty}(\omega,\, \bar{\sigma}^*)_{G_n}\\ &=\lim_{n\to\infty}\{-\int_{\partial G_n}w\sigma+\sum_{G_n}[\int_{A_i}\omega\int_{B_i}\sigma-\int_{B_i}\omega\int_{A_i}\sigma]\}\\ &=\lim_{n\to\infty}-\int_{\partial G_n}f\psi,\end{aligned}$$

where $w=\int\omega$ on $R-\{A_j,\, B_j\}$. Similarly we have

$$0=(\omega, \sigma^*)=-\lim_{n\to\infty}\int_{\partial G_n} f\bar{\psi}.$$

Lemma 2. If $f\in \mathrm{M}(X; 1/\delta_p)$ *is holomorphic in* R, *then* $\mathrm{d}f$ *is identically zero.*

Since $\mathrm{d}f$ belongs to $\mathrm{D}(X; 1/\delta_q)$, it follows that

$$(\mathrm{d}f, \mathrm{d}f)=(\mathrm{d}f, i\mathrm{d}f^*)=-i\lim_{n\to\infty}\int_{\partial G_n} f\overline{\mathrm{d}f}=0.$$

Lemma 3. (*cf.*[18]) *Let* K *be a field and* Y *and* Z *be two linear spaces over* K. *Suppose that* h *is a bilinear form defined on the product space* $Y\times Z$, *and that* Y_0 (*resp.* Z_0) *is the left* (*resp. right*) *kernel of* h, *that is*

$$Y_0=\{y\in Y;\ h(y, z)=0 \text{ for all } z\in Z\},$$
$$Z_0=\{z\in Z;\ h(y, z)=0 \text{ for all } y\in Y\}.$$

then $\dim Y/Y_0=\dim Z/Z_0$, where both left and right sides may be infinite. If at least one of the quotient spaces Y/Y_0 or Z/Z_0 is finite dimensional, we have an isomorphism $Y/Y_0\simeq Z/Z_0$.

Proposition 2. (*Riemann-Roch theorem*)

$$\dim\frac{\mathrm{M}(X; 1/\delta_p)}{\mathrm{S}(X; 1/\delta)}=\dim\frac{\mathrm{D}(X; 1/\delta_q)}{\mathrm{D}(X; \delta)}$$

where both left and right sides may be infinite.

If δ_p *is a finite divisor,*

$$\dim \mathrm{S}(X; 1/\delta)=[\mathrm{ord}\ \delta_p+1-\min(\mathrm{ord}\ \delta_q, 1)]-\dim\frac{\mathrm{D}(X; 1/\delta_q)}{\mathrm{D}(X; \delta)}.$$

Proof. We consider a bilinear form defined on the product space $\mathrm{M}(X; 1/\delta_p)\times\mathrm{D}(X; 1/\delta_q)$

$$h(f, \psi)=2\pi i\lim_{n\to\infty}\sum_{G_n}\operatorname*{Res}_{p_j} f\psi.$$

Since ψ is regular at each p_j, additive constants (including periods) of f have no effect for the residue of $f\psi$ at p_j. So we regard f as a function on $R'=R-\{A_j, B_j\}$. For any $f\in\mathrm{M}(X; 1/\delta_p)$ and any $\psi\in\mathrm{D}(X; 1/\delta_q)$, there exist a region G_m and differentials ω and σ in X such that $\mathrm{d}f=\omega$ and $\psi=\sigma$ on $R-G_m-G$. Then

$$\begin{aligned}h(f, \psi)&=2\pi i\lim_{n\to\infty}\sum_{G_n}\operatorname*{Res}_{p_j} f\psi\\&=2\pi i\lim_{n\to\infty}\{\sum_{G_n}\operatorname*{Res}_{x} f\psi-\sum_{G_n}\operatorname*{Res}_{q_i} f\psi\}\\&=\lim_{n\to\infty}\left\{\int_{\partial G_n} f\psi-\sum_{G_n}\left[\int_{A_i}\mathrm{d}f\int_{B_i}\psi-\int_{B_i}\mathrm{d}f\int_{A_i}\psi\right]\right\}-2\pi i\sum\operatorname*{Res}_{q_i} f\psi\\&=-\sum_{G_m}\left[\int_{A_i}\mathrm{d}f\int_{B_i}\psi-\int_{B_i}\mathrm{d}f\int_{A_i}\psi\right]-2\pi i\sum\operatorname*{Res}_{q_i} f\psi.\end{aligned}$$

Hence $h(f, \psi)$ has a finite complex value and h is a well defined bilinear form. If f belongs to $\mathrm{S}(X; 1/\delta)$, for any $\psi\in\mathrm{D}(X, 1/\delta_q)$ $f\psi$ has poles only at $\{p_j\}$. We have

$$h(f, \psi)=-\sum\left[\int_{A_i} df \int_{B_i} \psi-\int_{B_i} df \int_{A_i} \psi\right]=0.$$

The function f belongs to the left kernel of h and $S(X; 1/\delta)$ is contained in the left kernel of h. Conversely if f belongs to the left kernel of h,

$$0=h(f, \psi_{A_j})=\int_{A_j} df \int_{B_j} \psi_{A_j}-\int_{B_j} df \int_{A_j} \psi_{A_j}.$$

It follows that $\int_{B_j} df=\int_{A_j} df=0$ for any j, and f is a single valued meromporphic function in $M(X; 1/\delta_p)$.
If $\delta=\delta_p$, then f belongs to $S(X; 1/\delta)$. If $\delta\not\equiv\delta_p$, then

$$0=h(f, \psi_{q_1,q_i})=-2\pi i\,(\operatorname*{Res}_{q_1} f\,\psi_{q_1,q_i}+\operatorname*{Res}_{q_i} f\,\psi_{q_1,q_i})=2\pi i\,(f(q_i)-f(q_1)).$$

Hence $f(q_i)=f(q_1)$ for any i. Further, since

$$0=h(f, \psi_{q_i,k})=-2\pi i \operatorname*{Res}_{q_i} f\,\psi_{q_i,k} \text{ for } k\leq\nu_i-1,$$

the function $f-f(q_1)$ has at least ν_i-ple zeros at q_i. Above all we know that $S(X; 1/\delta)$ is the left kernel of h. Next let $f\in M(X; 1/\delta_p)$ and $\psi\in D(X; \delta)$. Then $f\psi$ is regular analytic at $\{p_j\}$. Hence $h(f, \psi)=0$. Multi-valued meromorphic functions $\Psi_{p_j,k}=\int\psi_{p_j,k}$ $(1\leqq k\leqq\mu_j)$ belong to $M(X; 1/\delta_p)$. When ψ belongs to the right kernel of h, we have

$$0=h(\psi_{p_j,k}, \psi)=2\pi i \operatorname*{Res}_{p_j} \Psi_{p_j,k}\,\psi.$$

Therefore ψ has at least μ_j-ple zeros at p_j. Thus $D(X; \delta)$ is the right kernel of h. By Lemma 3 it follows that

$$\dim \frac{M(X; 1/\delta_p)}{S(X; 1/\delta)}=\dim \frac{D(X; 1/\delta_q)}{D(X; \delta)}.$$

If δ_p is a finite divisor $p_1^{\mu_1}p_2^{\mu_2}\cdots p_n^{\mu_n}$, it is easily seen that $\{\Psi_{p_j,k}\}_{1\leq j\leq n,1\leq k\leq\mu_j}$ span $M(X; 1/\delta_p)$ $(\delta_q\neq1)$, and a constant 1 and $\{\psi_{p_j,k}\}_{1\leq j\leq n,1\leq k\leq\mu_j}$ make a basis of $M(X; 1/\delta_p)$ $(\delta_q=1)$. We find that $\dim M(X; 1/\delta_p)=\operatorname{ord}\delta_p+1-\min(\operatorname{ord}\delta_q, 1)$. We can obtain the conclusion:

$$\dim S(X; 1/\delta)=[\operatorname{ord}\delta_p+1-\min(\operatorname{ord}\delta_q, 1)]-\dim \frac{D(X; 1/\delta_q)}{D(X; \delta)}.$$

We remark that $D(X; 1)$ is spanned by $\{\psi_{A_j}\}_{b_j\neq0}$ and $\{\psi_{B_j}\}_{b_j=0}$, and $D(X; 1/\delta_q)$ is spanned by $\{\psi_{A_j}\}_{b_j\neq0}$, $\{\psi_{B_j}\}_{b_j=0}$, $\{\psi_{q_i,k}\}_{1\leq k\leq\nu_i-1}$ and $\{\psi_{q_i,q_1}\}_{2\leq i\leq m}$ if $\delta_q=q_1^{\nu_1}q_2^{\nu_2}\cdots q_m^{\nu_m}$. Thus when the genus g of R is finite and δ_q is a finite divisor, we have

$$\dim D(X; 1/\delta_q)=\begin{cases} g & \text{if } \delta_q=1 \\ g+\operatorname{ord}\delta_q-1 & \text{if } \delta_q\neq1. \end{cases}$$

Corollary 2.1. Let R be a Riemann surface with a finite genus g and $\delta=\delta_p/\delta_q$ is a finite divisor. Then it holds that

$$\dim S(X; 1/\delta)=\dim D(X; \delta)+\operatorname{ord}\delta-g+1.$$

2.3 *Abel's theorem*

Let δ_p and δ_q be finite or infinite divisors on R whose supports are disjointed from each other and are contained in $\bigcup_i V_{i,1/2}$ ($V_{i,1/2}=\{z_i; |z_i|<1/2\}$). Suppose that the restrictions to each $V_{i,1/2}$ of δ_p, δ_q have the same finite degree. Write them as $p_{i,1}p_{i,2}\cdots p_{i,n}$, $q_{i,1}q_{i,2}\cdots q_{i,n}$, where $p_{i,j}$ (resp. $q_{i,j}$) may coincide with $p_{i,k}$ (resp. $q_{i,k}$) for $j\neq k$. Further we assume that there exists a closed C^1-differential θ in $R-\bigcup_{i,j,k}(p_{i,k}\cup q_{i,j})$ such that

$$\text{(i)}\quad \theta=\begin{cases} d[\sum_j \log(z_i-p_{i,j})-\log(z_i-q_{i,j})] & \text{on } V_{i,1/2} \\ 0 & \text{on } R-G \end{cases}$$

$$\text{(ii)}\quad (\theta,\theta)_{G-\bigcup_i V_{i,1/2}}<\infty.$$

Under these circumstances we can formulate Abel's theorem.

Proposition 3. Abel's theorem

The following two conditions are equivalent.

(1) *There exists a single-valued meromorphic function f such that*

(i) *the divisor of f is δ,* (ii) d $\log f$ *has an X-behavior.*

(iii) $a_i\int_{A_i} d\arg f=b_i\int_{B_i} d\arg f$ *for any i.*

(2) *Let C be a chain which consists of chains C_1 in a G_m and C_2 in $G\cap(R-G_m)$ and let* $\partial C_1=\sum_{G_m}(p_{i,j}-q_{i,j})$, $\partial C_2=\sum_{R-G_m}(p_{i,j}-q_{i,j})$, *then*

$$\int_C \psi_{A_j} \text{ and } \int_C \psi_{B_j} \text{ are all integers.}$$

Proof. Let $R'=R-\{A_j, B_j\}$, and $G_n'=G_n\cap R'$. We put $\Psi_{A_j}=\int\psi_{A_i}$ and $\Psi_{B_j}=\int\psi_{B_j}$ in R'. If f is a meromorphic function in (1), then

$$\begin{aligned}\int_C\psi_{A_j}&=\lim_{n\to\infty}\sum_{G_n}\text{Res}\,\Psi_{A_j}\,d\log f\\ &=\frac{1}{2\pi i}\lim_{n\to\infty}\left\{\int_{\partial G_n}\Psi_{A_j}\,d\log f-\sum_{G_n}\left[\int_{A_i}\psi_{A_j}\int_{B_i}d\log f-\int_{B_i}\psi_{A_j}\int_{A_i}d\log f\right]\right\}\\ &=-\frac{1}{2\pi}\sum\left\{\int_{A_i}\psi_{A_j}\int_{B_i}d\arg f-\int_{B_i}\psi_{A_j}\int_{A_i}d\arg f\right\}\\ &=\frac{1}{2\pi}\int_{A_j}d\arg f.\end{aligned}$$

Similary $\int_C\psi_{B_j}=\frac{1}{2\pi}\int_{B_j}d\arg f$. Hence these are all integers. Conversely we suppose that (2) is filled. Now remark that there exists a closed differential $\tilde\theta$ such that $\tilde\theta=\theta^*$ on $R-G\cup(\bigcup_i V_{i,1/2})$. Let ω (resp. τ) be the orthogonal projection to $X=\Gamma_x+\Gamma_{eo}$ (resp. $\Gamma_x+\Gamma_{eo}{}^*$) of $\theta+\tilde\theta^*$. Then $\phi'=\theta-\omega=\tau-\tilde\theta^*$ is closed and coclosed. Hence ϕ' is harmonic in $R-\{p_{i,j}, q_{i,j}\}$. Let $\phi=(\phi'+i\phi'^*)/2$. Then ϕ coincides with $(-\omega+i\tau^*)/2\in\Gamma_x+\Gamma_{eo}$ on $R-G$. Thus ϕ is a meromorphic differential with X-behavior which satisfies $a_i\int_{A_i}\phi=b_i\int_{B_i}\phi$ for any i. So we can obtain

$$\frac{1}{2\pi i}\int_{A_j}\phi=\lim_{n\to\infty}\sum_{G_n}\text{Res }\Psi_{A_j}\,\phi=\int_C\psi_{A_j}\text{ and }\frac{1}{2\pi i}\int_{B_j}\phi=\int_C\psi_{B_j}.$$

It follows that $f(p)=\exp\int^p\phi$ is a single-valued meromorphic function and has the required properties. The function in (1) is uniquely determined except for multiplicative constants (cf. Lemma 2).

3. The case of finite bordered Riemann surfaces

3.1 *ME-behavior*

Let R be a finite bordered Riemann surface with a finite genus g whose boundary ∂R consists of a finite number of compact analytic curves $\{C_i\}$. We denote by $V_i=\{1/2<|z_i|<1\}$ ring domains each of whose boundary $\{z_i;|z_i|=1\}$ is C_i. Write $z_i=r_i(\cos\theta_i+i\sin\theta_i)$. Then a harmonic differential can be written as

$$\omega=\tilde{c}_0{}^i\,\mathrm{d}\log r_i+d_0{}^i\,\mathrm{d}\theta_i$$
$$+\sum_n\{c_n{}^i\,\mathrm{d}(r_i{}^n\cos n\theta_i+d_n{}^i\,\mathrm{d}(r_i{}^n\sin n\theta_i)\}$$

in each V_i. We shall denote $c_n{}^i$ and $d_n{}^i$ for ω by $c_n{}^i(\omega)$ and $d_n{}^i(\omega)$. Let

$\Gamma_E=\{\omega;\ \omega$ be a harmonic differential in Γ_h such that

(i) $d_0{}^i(\omega)=0$ for any i,

(ii) $d_m{}^i(\omega)=d^i_{-m}(\omega)$ for any m and i,

(iii) $a_i\int_{A_i}\omega=b_i\int_{B_i}\omega$ for any $i,\}$,

where $\{a_i, b_i\}$ is a real number system such that $|a_i|+|b_i|\neq 0$. Now we show that Γ_E is a behavior space. By the condition (i) Γ_E is a subspace of Γ_{hse}. For a differential ω is Γ_E the integral $w=\int\omega$ in $R'=R-\{A_j, B_j\}$ can be regarded as a cosine function with respect to θ_i on each boundary component C_i, because w is a harmonic function with a finite Dirichlet integral in each V_i and ω satisfies the condition (ii). Let ω and σ belong to Γ_E. Then we have

$$(\omega,\sigma^*)=-\int_{\partial R}w\bar{\sigma}+\sum\left[\int_{A_j}\omega\int_{B_j}\bar{\sigma}-\int_{B_j}\omega\int_{A_j}\bar{\sigma}\right]$$
$$=\lim_{r_i\to 1}-\sum_i\Big\{\int_{|z_i|=r_i}(\tilde{c}_0{}^i(\omega)\log r_i+\sum_n c_n{}^i(\omega)\,r_i{}^n\cos n\theta_i$$
$$+\sum_m d_m{}^i(\omega)\,r_i{}^m\sin m\theta_i)\,(\overline{\tilde{c}_0{}^i(\sigma)}\,\mathrm{d}(\log r_i)+\sum_j\overline{c_j{}^i(\sigma)}\,\mathrm{d}(r_i{}^j\cos j\theta_i)$$
$$+\sum_k\overline{d_k{}^i(\sigma)}\,\mathrm{d}(r_i{}^k\sin k\theta_i))\Big\}$$
$$=-\sum_i\Big\{\int_{C_i}\Big\{\sum_{n,k}\frac{k}{2}c_n{}^i(\omega)\,\overline{d_k{}^i(\sigma)}\,[\cos(n+k)\,\theta_i+\cos(n-k)\,\theta_i]$$
$$+\sum_{m,j}\frac{j}{2}d_m{}^i(\omega)\,\overline{c_j{}^i(\sigma)}\,[\cos(m+j)\,\theta_i-\cos(m-j)\,\theta_i]\Big\}\mathrm{d}\theta_i\Big\}$$

$$=-\pi \sum_i \{ \sum_n (nc_n^i(\omega)\,\overline{d_n^i(\sigma)}-nc_n^i(\omega)\,\overline{d_{-n}^i(\sigma)})$$

$$+\sum_j (jd_{-j}^i(\omega)\,\overline{c_j^i(\sigma)}-jd_j^i(\omega)\,\overline{c_j^i(\sigma)})\}$$

$$=0.$$

Hence Γ_E is orthogonal to $\Gamma_E{}^*$. If a differential ω in Γ_{ho} satisfies that $a_i\int_{A_i}\omega=b_i\int_{B_i}\omega$ for any i, then ω belongs to Γ_E. Hence $\Gamma_{hm}\subset\Gamma_E$ and $\Gamma_E{}^{\perp *}\subset\Gamma_{hse}$. Further there exists non-zero $\omega\in\Gamma_{ho}$ ~~$\cap\Gamma_{ho}{}^*$~~ $\cap\Gamma_{he}$ such that $a_j\int_{A_j}\omega=b_j\int_{B_j}\omega$ and $\int_{A_i}\omega=\int_{B_i}\omega=0$ for $i\neq j$. For $\sigma\in\Gamma_E{}^{\perp *}$

$$0=(\omega,\bar{\sigma}^*)=\int_{A_j}\omega\int_{B_j}\sigma-\int_{B_j}\omega\int_{A_j}\sigma$$

and we obtain $a_j\int_{A_j}\sigma=b_j\int_{B_j}\sigma$. Now the space $\Gamma_E{}^{\perp *}$ contains Γ_E and has an orthogonal decomposition $\Gamma_E{}^{\perp *}=\Gamma_E+\Gamma_E{}^{\perp *}\cap\Gamma_E{}^{\perp}$. Let $\omega\in\Gamma_E{}^{\perp *}\cap\Gamma_E{}^{\perp}$. If $\sigma=ds$ belongs to Γ_{he}, we have

$$(\sigma,\omega^*)=-\int_{\partial R}s\bar{\omega}$$

$$=-\sum_i\int_{C_i}(\tilde{c}_0^i(\sigma)\log r_i+\sum_n c_n^i(\sigma)\,r_i^n\cos n\theta_i+\sum_m d_m^i(\sigma)\,r_i^m\sin m\theta_i)$$

$$(\overline{\tilde{c}_0^i(\omega)}\,\mathrm{dlog}\, r_i+\sum_j\overline{c_j^i(\omega)}\,\mathrm{d}\,(r_i^j\cos j\theta_i)+\sum_k\overline{d_k^i(\omega)}\,\mathrm{d}\,(r_i^k\sin k\theta_i))$$

$$=-\sum\int_{C_i}\Big\{\sum_{n,k}\frac{k}{2}c_n^i(\sigma)\,\overline{d_k^i(\omega)}\,[\cos(n+k)\,\theta_i+\cos(n-k)\,\theta_i]$$

$$+\sum_{m,j}\frac{j}{2}d_m^i(\sigma)\,\overline{c_j^i(\omega)}\,[\cos(m+j)\,\theta_i-\cos(m-j)\,\theta_i]\Big\}\mathrm{d}\theta_i$$

$$=-\pi\sum_i\Big\{\sum_k\frac{k}{2}\overline{d_k^i(\omega)}\,[c_{-k}^i(\sigma)+c_k^i(\sigma)]$$

$$+\sum_j\frac{j}{2}\overline{c_j^i(\omega)}\,[d_{-j}^i(\sigma)-d_j^i(\sigma)]\Big\}.$$

We can choose an exact harmonic differential $\sigma=ds$ such that

(i) $d_{-j}^i(\sigma)=d_j^i(\sigma)$ for any i and j,

(ii) $c_{-k}^i(\sigma)=-c_k^i(\sigma)$ for any pair $(i,k)\neq(i_0,\pm k_0)$,

(iii) $c_{-k_0}^{i_0}(\sigma)+c_{k_0}^{i_0}(\sigma)=d_{k_0}^{i_0}(\sigma)$.

Since ds belongs to Γ_E, we obtain $d_{k_0}^{i_0}(\omega)=0$. Hence $d_k^i(\omega)=0$ for any i and k. Similarly we know that $c_k^i(\omega)=0$ for any i and k. Thus $\omega=0$ and $\Gamma_E=\Gamma_E{}^{\perp *}$. More generally we obtain the following in the same way.

Proposition 4. Let M_i be subsets of the natural numbers and
$\Gamma_{ME}=\{\omega$; *ω is a harmonic semiexact differential such that*

(i) $d_n^i(\omega)=d_{-n}^i(\omega)$ *for any pair* (i,n), $n\notin M_i$,

(ii) $c_m^i(\omega)=-c_{-m}^i(\omega)$ *for any pair* (i,m), $m\in M_i$,

(iii) $a_j\int_{A_j}\omega=b_j\int_{B_j}\omega$ *for any* $j.\}$.

Then the orthogonal complement of Γ_{ME} *in* Γ_h *is the space* $\Gamma_{ME}{}^*$ *and* Γ_{ME} *is a behavior space.*

Remark. For an integral $w=\int\omega$ of a differential ω in Γ_{ME} the boundary function has a Fourier expansion of the following type

$$w(\theta_i)=\sum_{m\notin M_i}c_m{}^i\cos m\theta_i+\sum_{n\in M_i}d_n{}^i\sin n\theta_i+c_0{}^i \text{ on } C_i.$$

Definition. A meromorphic function w *is said to have an ME-behavior if* $\mathrm{d}w$ *is equal to a differential in* $\Gamma_{ME}+\Gamma_{eo}$ *in some neighbourhood of the ideal boundary.*

It can be said that a meromorphic function w has an ME-behavior if and only if w has a finite Dirichlet integral in a neighbourhood of the ideal boundary and has a Fourier expansion of the above type on the boundary. In fact let $\mathrm{d}w$ be written as $\omega+\mathrm{d}f$ with $\omega\in\Gamma_{ME}$ and $\mathrm{d}f\in\Gamma_{eo}$ in a neighbourhood of the boundary ∂R. Then w has a Fourier expansion of the above type and f may vanish on ∂R. Conversely we suppose that w has a finite Dirichlet integral and has a Fourier expansion of the above type. There is a C^∞ function $\hat{w}$ such that $\hat{w}=w$ in each V_i. Let $\mathrm{d}\hat{w}=\mathrm{d}w_1+\mathrm{d}f_1$ with $\mathrm{d}w_1\in\Gamma_{he}$ and $\mathrm{d}f_1\in\Gamma_{eo}$, and

$$w_1(z_i)=\tilde{c}_0{}^i\log r_i+\sum_m c_m{}^i\, r_i{}^m\cos m\theta_i+\sum_n d_n{}^i\, r_i{}^n\sin n\theta_i$$

in each V_i. Then the Fourier expansion of w_1 on C_i is

$$w_1(\theta_i)=\sum_{m>0}(c_m{}^i+c_{-m}^i)\cos m\theta_i+\sum_{n>0}(d_n{}^i-d_{-n}^i)\sin n\theta_i+c_0{}^i.$$

Since f_1 is constant on ∂R, we can regard that the Fourier expansion of w_1 on C_i coincides with that of w. Hence $c_m{}^i=-c_{-m}^i$ for $m\in M_i$ and $d_n{}^i=d_{-n}^i$ for $n\notin M_i$. Thus $\mathrm{d}w_1\in\Gamma_{ME}$ and w has an ME-behavior. Next let a meromorphic differential $\sigma=\mathrm{d}s$ satisfy that $\sigma=\sigma_1+\mathrm{d}f$ with $\sigma_1\in\Gamma_{ME}$ and $\mathrm{d}f\in\Gamma_{eo}$ in a neighbourhood of the ideal boundary. Since $\sigma=i\sigma^*$, we have

$$c_m{}^j(\sigma)=-id_m{}^j(\sigma)=-c_{-m}^j(\sigma)=id_{-m}^j(\sigma)\quad\text{for } m\in M_j,$$

$$d_n{}^j(\sigma)=ic_n{}^j(\sigma)=d_{-n}^j(\sigma)=ic_n{}^j(\sigma)\quad\text{for } n\notin M_j.$$

It follows that

$$s(z_i)=\sum_{m\in M_i}c_m{}^i(\sigma)\,(z_i{}^m-1/z_i{}^m)+\sum_{n\notin M_i}c_n{}^i(\sigma)\,(z_i{}^n+1/z_i{}^n)+c$$

in each V_i. Note that $s(z_i)$ is defined in $\{z_i;\ \rho<|z_i|<1/\rho\}$ and converges in the sense of the Dirichlet norm. Thus we know that s is analytic on each boundary C_i with respect to z_i. As a consequence of this result,

Proposition 5. A meromorphic function with an ME-behavior is bounded in a neighbourhood of the ideal boundary, and further it is represented as follows:

$$\sum_{m\notin M_i} 2c_m^i \cos m\theta_i + \sum_{n\in M_i} 2d_n^i \sin n\theta_i + c$$

on each C_i.

Now E-behavior is the case that all M_i's are empty. As to E-behavior we have

Proposition 6. Let f be a meromorphic function with E-behavior and have n poles. If a complex number α does not belong to the boundary clustér set of f ($f(\partial R)$), the number of α points of f is n. The function f is a conformal mapping of R onto the covering surfaces of the complex plane which is n sheeted and have a finite number of slits represented as

$$\sum c_m^i \cos m\theta_i \quad (0\leq\theta_i<2\pi) \text{ on } C_i.$$

Proof. Since we can regard $f(z_i)=f(1/z_i)$, we have $1/(f(z_i)-\alpha)=1/f(1/z_i)-\alpha)$. Therefore the function $1/(f-\alpha)$ is also a meromorphic function with E-behavior. It follows from Lemma 1 that

$$n(f, \alpha)-n(f, \infty)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\partial G_n}\frac{df}{f-\alpha}=0,$$

where $n(f, \alpha)$ is the number of α points and $n(f, \infty)=n$. Thus by use of Proposition 5 we obtain the statement.

Corollary 6.1 If the genus g of R is positive, then $\dim D(E; p)\leq g-1$ *for any point* $p\in R$.

Proof. If $\dim D(E; p_0)=g$, by Corollary 2.1 there exists a meromorphic function $f\in S(E; 1/p_0)$. The Riemann surface R is conformal to a planar region by f. This is a contradiction.

Corollary 6.2 There exists a set of distinct points $p_1, p_2,\cdots, p_g$ such that $\dim D(E; p_1p_2\cdots p_g)=0$.

Proof. Note that (i) $\dim D(E; p_1)=g-1$, (ii) $\dim D(E; p_1p_2\cdots p_s)\geq g-s$, (iii) if a differential in $D(E; p_1p_2\cdots p_s)$ does not vanish at a point p_{s+1}, then $\dim D(E; p_1p_2\cdots p_sp_{s+1}) <\dim D(E; p_1p_2\cdots p_s)$.

Lemma 4 Let δ, δ' and δ'' be finite divisors. Let f, f_1 and ω belong to $S(E; \delta)$, $S(E; \delta')$ and $D(E; \delta'')$ respectively. Then ff_1 belongs to $S(E; \delta\delta')$ and $f\omega$ belongs to $D(E; \delta\delta'')$.

Proof. Since f and f_1 is bounded in V_i, ff_1 and $f\omega$ have finite Dirichlet integrals in V_i. By direct calculation they have Fourier expansions on each C_i which give E-behaviors.

Proposition 7. Let δ be a finite divisor. If a differential ω belongs to $D(E; \delta)$, *then* $\text{ord}\,(\omega)\leq 2g-2$, *where (ω) is the divisor of ω.*

Proof. Let $(\omega)=\delta'\delta''$ and $\dim D(E; \delta')=m$, $\dim D(E; \delta'')=n$. If $f\in S(E; 1/\delta')$, then $f\omega$ belongs to $D(E; \delta'')$. It follows that $\dim S(E; 1/\delta')\leqq n$ and similarly $\dim S(E; 1/\delta'')\leqq m$. Thus by Corollary 2.1, $m+n\geqq n+m+\text{ord}\,\delta'+\text{ord}\,\delta''-2g+2$. This proves the statement.

In the same way as in Lemma 4 we have:

Lemma 5. Let M_i be empty or all natural integers. Let s, s_1 and σ belong to $S(ME;$

δ), $\mathrm{S}(ME, \delta')$ *and* $\mathrm{D}(ME; \delta'')$ *respectively. Then* ss_1 *belongs to* $\mathrm{S}(E; \delta\delta')$ *and* $s\sigma$ *belongs to* $\mathrm{D}(E; \delta\delta'')$. *If a meromorphic function f in* $\mathrm{S}(ME; \delta)$ *has n poles, then the number of a points for* f^2 *is 2n except for* $a \in f(\partial R) \cap -f(\partial R)$.

3.2 *X-Weierstrass points*

Let the genus of Riemann surface R be a finite number g. Corresponding to classical Weierstrass points we define the following.

Definition. A point $p \in R$ *is called an X-Weierstrass point if there exists a meromorphic function with an X-behavior which has a pole of order n* ($\leq g$) *only at p. A positive integer m is called an X-gap value at p if no meromorphic function with an X-behavior has a pole of the order m only at p.*

Lemma 6. There exists a family of linearly independent holomorphic differentials with an X-behavior $\psi_1, \psi_2, \cdots, \psi_g$ *such that the order* μ_i *of zero of* ψ_i *at p satisfies*

$$0 \leq \mu_1 < \mu_2 < \cdots < \mu_g.$$

The numbers $\mu_1+1, \mu_2+1, \cdots, \mu_g+1$ *are X-gap values at p.*

Proof. If ψ_1 and ψ_2 has the same order of zero at p, then there exist complex numbers a and b such that ψ_1 and $\psi_2'=a\psi_1+b\psi_2$ have different orders of zero at p and $\psi_1, \psi_2', \cdots, \psi_g$ are linearly independent. This proves the first assertion. Suppose that f is a meromorphic function with an X-behavior which has a pole of order μ_i+1 only at p. Then by Lemma 2 we have

$$\operatorname*{Res}_{p} f\psi_i = \lim_{n\to\infty} \int_{\partial G_n} f\psi_i = 0.$$

This is a contradiction. Hence μ_i+1 is an X-gap value.

Remark. From Corollary 2.1 we know that the number of X-gap values at every point p is g. Hence $\mu_1+1, \mu_2+1, \cdots, \mu_g+1$ are all X-gap values at p.

Now we denote ψ_i of this Lemma as $\psi_i = f_i(z)\, dz$ and consider the Wronskian

$$W(z) = \begin{vmatrix} f_1(z) & \cdots & f_g(z) \\ f_1'(z) & \cdots & f_g'(z) \\ & \cdots & \\ f_1^{(g-1)}(z) & \cdots & f_g^{(g-1)}(z) \end{vmatrix}.$$

The $W(z)$ has a Taylor expansion and the first term is

$$(-1)^{(g-1)g/2} \prod_{j<k} (\mu_j - \mu_k)\, z^{\Sigma(\mu_i+1)-(1+2+\cdots+g)}.$$

We can assert the following.

Proposition 8. X-Weierstrass points coincide with the zeros of the Wronskian $W(z)$ *and they are at most a countable number of points which have no accumulating point in R.*

Corollary 8.1 (*i*) *ME--Weierstrass points are at most finite.* (*ii*) *If m is an E-gap value, then m is less than 2g.*

Proof. By Proposition 5 $W(z)$ is analytic on the boundary. This proves (i). As to (ii) we note Proposition 7 and dim $D(X; p^{2g-1})=0$.

Remark. The zeros of Wronskian are independent on the choice of the linearly independent holomorphic differentials with an X-behavior. Next as for an E-behavior, if the degree of $W(z)\, dz^{g(g+1)/2}$ is N, then the number of E Weierstrass points is less than N and is larger than $2N/g(g-1)$. The proof is analogous to the classical Hurwitz theorem.

3.3 *Jacobi's inversion problem*

Here we can formulate Jacobi's inversion problem on a finite bordered Riemann surface R with genus g. Let ψ_i be the holomorphic differential with an X-behavior. For complex numbers e_i and e_i' we denote $e_i \equiv e_i' \bmod P(\psi_i)$ if there exist integers m_j and m_j' such that $e_i' = \sum_{j=1}^{g} m_j \int_{A_j} \psi_i + \sum_{j=1}^{g} m_j' \int_{B_j} \psi_i + e_i$.

Peroposition 9. Let $\psi_1, \psi_2, \cdots, \psi_g$ be linearly independent holomorphic differentials with an E-behavior. For a point p_0 in R and complex numbers $e_1, e_2, \cdots, e_g$, there exist points $p_1, p_2, \cdots, p_g$ on R such that

$$\sum_{j=1}^{g} \int_{p_0}^{p_j} \psi_i \equiv e_i \bmod P(\psi_i) \text{ for } i=1, \cdots, g.$$

Proof. By Corollary 6.2 there exists a set of distinct points $q_1, \cdots, q_g$ $(\in R)$ such that dim $D(E; q_1 q_2 \cdots q_g)=0$. Take disjoint parametric discs $V_i=\{z_i; |z_i|<1\}$ about q_i and write $\psi_j = f_j(z_i)\, dz_i$ on V_i. Since

$$\begin{vmatrix} f_1(q_1) & \cdots & f_1(q_g) \\ & \cdots & \\ f_g(q_1) & \cdots & f_g(q_g) \end{vmatrix} \neq 0$$

we can assume that

$$u(p_1, p_2, \cdots, p_g) = \left(\sum_{j=1}^{g} \int_{q_j}^{p_j} \psi_1, \cdots, \sum_{j=1}^{q} \int_{q_j}^{p_j} \psi_g\right) \in C^g$$

is a homeomorphism from $V = V_1 \times V_2 \times \cdots \times V_g$ into a neighbourhood U in C^g, where integral $\int_{q_j}^{p_j} \psi_i$ is taken in V_j. There exist a positive integer n and points $\{p_j \in V_j\}_j$ such that $(e_1/n, \cdots, e_g/n) \in U = u(V)$ and $\sum_{j=1}^{g} \int_{q_j}^{p_j} \psi_i = e_i/n$ for $i=1, \cdots, g$. Let δ be a divisor $p_0^g (\tilde{p}_1 \tilde{p}_2 \cdots \tilde{p}_g)^n / (q_1 q_2 \cdots q_g)^n$. By Corollary 2.1

$$\dim S(E; 1/\delta) = \dim D(E; \delta) + 1 \geqq 1,$$

and there exists a non constant meromorphic function f in $S(E; 1/\delta)$. Since $(f)\delta$ is an integral divisor, there exist points $p_1, \cdots, p_r$ $(r \leqq g)$ in R such that $(f) = p_1 p_2 \cdots p_r (q_1 q_2 \ldots q_g)^n / p_0^g (\tilde{p}_1 \tilde{p}_2 \cdots \tilde{p}_g)^n$. By Proposition 6 for any $\varepsilon > 0$ there exists a complex number $\alpha \notin f(\partial R)$ such that $|\alpha| < \varepsilon$ and as for the divisor of $f - \alpha$

$$(f-a)=p_1(a)p_2(a)\cdots p_g(a)\sum_{k=1}^{g}\sum_{j=1}^{n}q_{kj}(a)/p_0{}^g\ (\tilde{p}_1\tilde{p}_2\cdots\tilde{p}_g)^n.$$

Since $f-a\neq 0$ on ∂R and $(f-a)\ (e^{i\theta_j})=(f-a)\ (e^{-i\theta_j})$ on C_j, the function $\log\ (f-a)$ is assumed to have single values in a neighbourhood of each boundary component. We can regard $\log\ (f-a)\ (z_j)=\log\ (f-a)\ (1/z_j)$ and know that $\mathrm{d}\log\ (f-a)$ has E-behavior. It follows that

$$\begin{aligned}\sum_{j=1}^{g}\int_{p_0}^{p_j(\alpha)}\psi_i+\sum_{k=1}^{g}\sum_{j=1}^{n}\int_{\tilde{p}_k}^{q_{kj}(\alpha)}\psi_i=\sum_{j=1}^{g}\int_{p_0}^{p_j(\alpha)}\psi_i+n\sum_{k=1}^{g}\int_{\tilde{p}_k}^{q_k}\psi_i+\beta(a)\\ =\sum \mathrm{Res}\ \Psi_i\ \mathrm{d}\log\ (f-a)\\ =\frac{1}{2\pi i}\lim_{n\to\infty}\Big\{\int_{\partial G_n}\Psi_i\ \mathrm{d}\log\ (f-a)\\ -\sum_{G_n}\Big[\int_{A_j}\psi_i\int_{B_j}\mathrm{d}\log\ (f-a)-\int_{B_j}\psi_i\int_{A_j}\mathrm{d}\log\ (f-a)\Big]\Big\}\\ =-\frac{1}{2\pi}\sum\Big[\int_{A_j}\psi_i\int_{B_j}\mathrm{d}\arg\ (f-a)-\int_{B_j}\psi_i\int_{A_j}\mathrm{d}\arg\ (f-a)\Big].\end{aligned}$$

We can assume that $(p_1(a), p_2(a), \cdots, p_g(a))$ converges to $(p_1, \cdots, p_g)$ on $\bar{R}\times\bar{R}\times\cdots\times\bar{R}$ and $\beta(a)$ converges to 0. Hence we have

$$\sum_{j=1}^{g}\int_{p_0}^{p_j}\psi_i\equiv e_i \bmod P(\psi_i)\ \text{for}\ i=1, \cdots, g.$$

Remark. A meromorphic differential with E-behavior on R can be regarded as a meromorphic differential on a compact Riemann surface $\tilde{R}$ which is obtained by sewing $e^{i\theta_j}$ with $e^{-i\theta_j}$ on each C_j of R. Hence the assertions for E-behavior correspond to classical theories on compact Riemann surfaces. It seems that Propositions 2 and 3 are obedient generalizations of classical Riemann-Roch and Abel's theories. For the sake of investigation of open Riemann surfaces, it is important to show the existence of behavior spaces as E-behavior spaces on arbitrary Riemann surfaces.

4. Examples

Example 1. Let D be a unit disc $\{z; |z|<1\}$ and Γ_E be the E-behavior space defined by the variable z. A meromorphic function $f(z)=z+1/z$ has an E-behavior and has a simple pole at zero. This is a well known slit mapping. Each slit mapping on a finite bordered Riemann surface whose slits are analytic curves has an E-behavior if the variables which define Γ_E are chosen suitably.

Example 2. Let $\{D_i\}_{1<i<\infty}$ be unit discs and x_n, y_n (<1) be increasing sequences of real positive numbers which converge to 1. Now slits $[-x_{2n+1}, -x_{2n}]$ and $[x_{2n}, x_{2n+1}]$ in D_{2n} and D_{2n+1} are sewn into the upper edges of D_{2n} with the lower edges of D_{2n+1} and the lower edges of D_{2n} with the upper edges of D_{2n+1}. Similarly $[-iy_{2n}, -iy_{2n-1}]$ and $[iy_{2n-1}, iy_{2n}]$ are sewn between D_{2n-1} and D_{2n}. By these sewings of $\{D_i\}$ we obtain a covering surface R on $\{z; z<1\}$. Then we can define a behavior space Γ_{ME} on R by the variable z in the same way as in a finite bordered Riemann surface. We can show that

$$(\omega, \sigma^*)=\sum -\int_{\partial D_i} w\bar{\sigma} \quad \text{for } \omega=\mathrm{d}w,\ \sigma\in\Gamma_{ME},$$

$$(\sigma, \omega^*)=\sum -\int_{\partial D_i} s\bar{\omega} \quad \text{for } \sigma=\mathrm{d}s\in\Gamma_{he}, \text{ and } \omega\in\Gamma_{ME}{}^{\perp *}.$$

We can obtain $\Gamma_{ME}=\Gamma_{ME}{}^{\perp *}=\bar{\Gamma}_{ME}$. Let $V_i=\{z;|z|<r_i\}$ be a disc in D_i which does not meet the branch points. Let $f(\tilde{z})=z+1/z$ on R, where z is the natural projection of $\tilde{z}$. The $f(\tilde{z})$ is a meromorphic function with an E-behavior which has infinite number of poles if $\sum_i$ area of $C-f(V_i)<\infty$ (cf. Proposition 2). For a cycle γ, the winding number of $f(\gamma)$ for $a\notin[-1, 1]$ is the period of $-i\mathrm{d}\log(f-a)$ with respect to γ. If we choose a suitable homology basis and $\{a_i, b_i\}$, then $\mathrm{d}\log(f-a)$ becomes a meromorphic differential with an E-behavior which has an infinite number of non-vanishing periods (cf. Proposition 3).

5. Remark

M. Yoshida defined ${}_R\Gamma_x$-behavior as follows:
A single-valued real harmonic function u defined in a neighbourhood of the ideal boundary of the Riemann surface R has ${}_R\Gamma_x$-behavior if $\mathrm{d}u$ and $\mathrm{d}u^*$ admit the following representations in a neighbourhood of the ideal boundary;

(i) $\mathrm{d}u=\mathrm{d}u_x+\mathrm{d}f_0$ with $\mathrm{d}u_x\in{}_R\Gamma_x$ and $\mathrm{d}f_0\in\Gamma_{e0}$,

(ii) $\mathrm{d}u^*=\omega_x+\omega_0$ with $\omega_x\in{}_R\Gamma_x{}^{\perp}$ and $\omega_0\in\Gamma_{e0}$,

where ${}_R\Gamma_x{}^{\perp}$ is the orthogonal complement of ${}_R\Gamma_x$ in the real Hilbert space ${}_R\Gamma_h$ which consists of real harmonic differentials in Γ_h. Now let f be a single valued-meromorphic function on R whose differential $\mathrm{d}f$ is distinguished[3]. Then the real and imaginary parts of f have ${}_R\Gamma_{hm}$-behaviors. The f is not always trivial if $R\in O_{HD}-O_G$ but it reduces to a constant if $R\notin O_{KD}$. Similarly it was expected that a meromorphic function whose real and imaginary parts have ${}_R\Gamma_x$-behavior reduces to a constant if $R\notin O_{KD}$. We have a counter example. That is to say, $S(ME; 1/\delta)$ contains a non-constant meromorphic function if the degree of δ is large enough and a function in $S(ME; 1/\delta)$ has $\Gamma_{ME}\cap{}_R\Gamma_{he}$-behavior. At last we note that (i) if $\Gamma_x=\Gamma_x{}^{\perp *}=\bar{\Gamma}_x$, then $\Gamma_x=\Gamma_x\cap{}_R\Gamma_h+i(\Gamma_x\cap{}_R\Gamma_h)$ and ${}_R\Gamma_h=\Gamma_x\cap{}_R\Gamma_h+\Gamma_x{}^*\cap{}_R\Gamma_h$, (ii) $\Gamma_{ME}\cap{}_R\Gamma_{he}$ consists of real valued harmonic functions $\{f\}$ each of which has a representation:

$$f(\theta_i)=\sum_{m\notin M_i} c_m{}^i\cos m\theta_i+\sum_{n\in M_i} d_n{}^i\sin n\theta_i+c_0{}^i \text{ on each } C_i.$$

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