Smoothness of rank M scaling functions

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This article investigates the smoothness of rank M scaling functions and gives some numerical results. Smoothness of a scaling function φ is studied by introducing an exponent $s_p(\varphi)$, which is represented by the spectral radius of the transfer operator associated with the reduced symbol of φ. A direct method for estimating the spectral radius is also proposed and used to obtain quite sharp estimates for $s_p(\varphi)$ and for the Hölder exponents. In fact, numerical experiments on $s_1(\varphi)$ for a certain class of scaling functions give estimates for their Hölder exponents much better than those obtained by the Sobolev estimates, which are known so far, combined with the Sobolev imbedding theorem.

Key Words: scaling functions; wavelets; Sobolev regularity; smoothness; transfer operator

1. INTRODUCTION

Smoothness of rank M scaling functions and the associated wavelets, with N vanishing moments and finite length, have been studied extensively by many authors ([1], [2], [3], [6], [8], [12], et. al.). As studied by these authors, smoothness of these wavelets is reduced to that of scaling functions. In this paper we study the smoothness of scaling functions in terms of s_p-exponent, defined below, and give some numerical results for the s_1-exponents.

For $p > 0$, the s_p-*exponent* of $f \in L^2(\mathbf{R})$ is defined by

$$s_p(f) := \sup \left\{ s \in \mathbf{R} : \int_{-\infty}^{\infty} |(1 + |\xi|)^s \hat{f}(\xi)|^p \, d\xi < \infty \right\},$$

which is known as the usual Sobolev exponent of f in case $p = 2$, where $\hat{f}$ denotes the Fourier transform:

$$\hat{f}(\xi) := \operatorname*{l.i.m.}_{N \to \infty} \int_{-N}^{N} f(x) e^{i\xi x} \, dx.$$

[1] Correspondence may be sent to Ôkura

For each $n \in \mathbf{N} \cup \{0\}$ let $\mathcal{C}^n$ denote the set of all n-times continuously differentiable functions on $\mathbf{R}$. For $\alpha = n + \sigma$ with $n \in \mathbf{N} \cup \{0\}$, $0 < \sigma < 1$, let

$$\mathcal{C}^\alpha := \left\{ f \in \mathcal{C}^n : \sup_{x \neq y} \frac{|f^{(n)}(x) - f^{(n)}(y)|}{|x-y|^\sigma} < \infty \right\},$$

where $f^{(n)}$ denotes the n-th derivative of f. The *Hölder exponent* of a function f is defined by

$$\alpha(f) := \sup\{\alpha \geq 0 : f \in \mathcal{C}^\alpha\}$$

with the convention that $\alpha(f) := 0$ if $f \notin \mathcal{C}^0$. It is well known that $s_2(f) - 1/2 \leq \alpha(f)$ for any $f \in L^2(\mathbf{R})$ by the Sobolev imbedding theorem. Moreover, if $f \in L^2(\mathbf{R})$ has compact support, then it holds (see Proposition 4.1) that

$$\left. \begin{cases} p s_p(f) & (0 < p \leq 1) \\ s_p(f) - \dfrac{p-1}{p} & (p \geq 1) \end{cases} \right\} \leq s_1(f) \leq \alpha(f) \leq s_r(f) \quad (r \geq 2). \tag{1.1}$$

Throughout this article we denote by $\varphi_{M,N,L}$ any rank M orthogonal scaling function of maximal degree N and length L. It is known that $L \geq MN$ and that there always exists an orthogonal scaling function $\varphi_{M,N,L}$ with $L = MN$. The latter is called a scaling function of minimal length and denoted by $\varphi_{M,N}$.

There arise two types of problem:

(1) Estimate the Hölder exponents and the s_p-exponents of $\varphi_{M,N}$.

(2) Find smoother scaling functions $\varphi_{M,N,L}$ of non minimal length.

As for problem (1), I. Daubechies and J. C. Lagarias [2], [3] used "the direct method" to obtain precise estimates for the Hölder exponents $\alpha(\varphi_{2,N})$ for small N and among other things they showed that $\varphi_{2,3} \in \mathcal{C}^1$. On the other hand, the following explicit expression of $s_2(\varphi_{M,N,L})$ was given in [4], [15] (see also [1]) for rank 2 and in [6] for general rank M:

$$s_2(\varphi_{M,N,L}) = N - \frac{1}{2} \log_M \rho(T_{|Q|^2}). \tag{1.2}$$

Here $Q(\xi)$ is the reduced symbol of the scaling function $\varphi_{M,N,L}$ and $\rho(T_{|Q|^2})$ is the spectral radius of the transfer operator $T_{|Q|^2}$ (see §§2 and 3 for definitions). Since $|Q|^2$ is a polynomial of $\cos\xi$, the operator $T_{|Q|^2}$ has a finite-dimensional invariant subspace spanned by $\cos k\xi$'s. It can be shown that the spectral radius $\rho(T_{|Q|^2})$ is identified with the largest eigenvalue of the positive matrix representing the restriction of $T_{|Q|^2}$ to this subspace. These observations on $s_2(\varphi_{M,N})$ lead to numerical results $\varphi_{2,7} \in \mathcal{C}^2$, $\varphi_{2,11} \in \mathcal{C}^3$, $\varphi_{3,9} \in \mathcal{C}^1$, for example. Some upper and lower bounds and asymptotic formulae for $s_2(\varphi_{M,N})$ as $N \to \infty$ are also given in [6]. Some related estimates for $s_p(\varphi_{M,N})$ $(p > 0)$ can be found in [14].

As for problem (2), the regularity of $\varphi_{M,N,L}$ can be remarkably improved even with small additional length $L - MN$: P. N. Heller and R. O. Wells, Jr. [6] clarified this fact by evaluating the s_2-exponents of the scaling functions whose reduced symbols vanish at periodic or preperiodic points of the map $\xi \mapsto M\xi \bmod 2\pi$.

In this paper, the s_p-exponents of scaling functions are studied. To this end, we use the transfer operator $T_{|Q|^p}$ instead of $T_{|Q|^2}$. Our main theorem enables us to estimate the s_p-exponents by using the maximum and the minimum of the function $T_{|Q|^p}(f)/f$ for a suitable function $f > 0$. Especially, in case $|Q| > 0$, the s_p-exponents can be represented in terms of the spectral radius $\rho(T_{|Q|^p})$ of the operator $T_{|Q|^p}$:

$$s_p(\varphi_{M,N,L}) = N - \frac{1}{p} \log_M \rho(T_{|Q|^p}). \tag{1.3}$$

Our results also cover some cases where $|Q|$ has zeros. See §4 for the details. Section 6 includes some numerical estimates of the s_1-exponents for the following families of scaling functions: (i) $\varphi_{M,N}$; (ii) $\varphi_{2,N,2N+4}$ with $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = 2\pi/3, 3\pi/4, 4\pi/5, 5\pi/6$; (iii) $\varphi_{3,N,3N+2}$ with $r(\cos\pi) = 0$; and (iv) $\varphi_{3,N,3N+3}$ with $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \pi/2, 3\pi/4, 5\pi/6, \pi$, where $r(x)$ denotes the polynomial satisfying $|Q(\xi)|^2 = r(\cos\xi)$. Although our s_1-estimates, including $s_1(\varphi_{2,3}) \doteqdot 0.980$, could not recover the above mentioned result $\varphi_{2,3} \in \mathcal{C}^1$ by I. Daubechies and J. C. Lagarias, we can pick out some outstanding results in our numerical experiments:

(i) $\varphi_{2,6} \in \mathcal{C}^2$, $\varphi_{2,8} \notin \mathcal{C}^3$, $\varphi_{2,9} \in \mathcal{C}^3$, $\varphi_{2,12} \notin \mathcal{C}^4$, $\varphi_{2,13} \in \mathcal{C}^4$;

(ii) $\varphi_{3,4} \in \mathcal{C}^1$, $\varphi_{3,77} \notin \mathcal{C}^2$, $\varphi_{3,78} \in \mathcal{C}^2$;

(iii) $\varphi_{2,6,16} \in \mathcal{C}^3$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = 5\pi/6$ or $4\pi/5$; $\varphi_{2,9,22} \in \mathcal{C}^4$ if $r(\cos 3\pi/4) = r'(\cos 3\pi/4) = 0$; $\varphi_{2,13,30} \in \mathcal{C}^5$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = 3\pi/4$ or $2\pi/3$;

(iv) $\varphi_{3,3,11} \in \mathcal{C}^1$, $\varphi_{3,6,20} \in \mathcal{C}^2$, $\varphi_{3,9,29} \in \mathcal{C}^3$, $\varphi_{3,13,41} \in \mathcal{C}^4$ if $r(\cos\pi) = 0$;

(v) $\varphi_{3,2,9} \in \mathcal{C}^1$ if $r(\cos 5\pi/6) = r'(\cos 5\pi/6) = 0$; $\varphi_{3,5,18} \in \mathcal{C}^2$ if $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = 5\pi/6$ or π; $\varphi_{3,8,27} \in \mathcal{C}^3$, $\varphi_{3,11,36} \in \mathcal{C}^4$ if $r(\cos\pi) = r'(\cos\pi) = 0$.

This article is organized as follows: In §2 we recall fundamental notions and facts on scaling functions and also give some supplementary results on Cohen's condition and the existence of scaling functions of special type treated in the numerical experiments. Section 3 includes some basic results on the transfer operators used in §§4 and 5. In §4 we give the main results of this article mentioned above. In §5 two theorems on comparison and approximation for the s_p-exponents by step functions are given. Section 6 contains the numerical results on the s_1-exponents. Finally, in Appendices A and B, we give the proof of the supplementary results in §2 and give some related examples of scaling functions.

2. FUNDAMENTALS ON RANK M SCALING FUNCTIONS

Throughout this article M denotes an integer satisfying $M \geq 2$. First we recall fundamental facts on rank M orthogonal scaling functions [1], [5], [6], [12], [8].

2.1. Rank M scaling functions

By a *rank M scaling function* we mean a non-trivial function $\varphi(x)$ in $L^2(\mathbf{R})$ satisfying the "scaling relation":

$$\varphi(x) = \sum_{k \in \mathbf{Z}} a_k \varphi(Mx - k) \tag{2.1}$$

for some real sequence $\{a_k\}_{k\in\mathbf{Z}}$ with $\sum_{k\in\mathbf{Z}} a_k = M$. The sequence $\{a_k\}_k$ is called the *scaling sequence* and the Fourier series $A(\xi) := \frac{1}{M}\sum_k a_k e^{ik\xi}$ is called the *symbol* of $\varphi(x)$. We note that $A(0) = 1$ and the relation (2.1) is equivalent to the following equation:

$$\hat{\varphi}(\xi) = A(\xi/M)\hat{\varphi}(\xi/M). \tag{2.2}$$

The *length* L of a scaling function φ is defined by $L := k_1 - k_0 + 1$, where $k_1 := \max\{k \in \mathbf{Z} :\ a_k \neq 0\}$ and $k_0 := \min\{k \in \mathbf{Z} :\ a_k \neq 0\}$. A scaling function $\varphi(x)$ has a finite length if and only if $\varphi(x)$ has compact support. In this case, we have

$$\hat{\varphi}(\xi) = \hat{\varphi}(0)\prod_{j=1}^{\infty} A(\xi/M^j). \tag{2.3}$$

A scaling function $\varphi(x)$ is said to be *orthogonal* if the sequence $\{\varphi(x-k)\}_{k\in\mathbf{Z}}$ is an orthogonal system in $L^2(\mathbf{R})$, which leads to the so called "orthogonality condition" [6] for symbol $A(\xi)$:

$$\sum_{k=0}^{M-1} |A(\xi + 2k\pi/M)|^2 = 1. \tag{2.4}$$

Throughout this article we only consider scaling functions of finite length with their symbols satisfying (2.4). We say that a scaling function $\varphi(x)$ is of *degree* N ($N \geq 1$) if

$$A(\xi) = \left(\frac{1 + e^{i\xi} + \cdots + e^{i(M-1)\xi}}{M}\right)^N Q(\xi) = \left(\frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})}\right)^N Q(\xi) \tag{2.5}$$

for some trigonometric polynomial $Q(\xi)$. If, in addition, $\varphi(x)$ is not of degree $N+1$, then N and $Q(\xi)$ in (2.5) are called the *maximal degree* and the *reduced symbol* of $\varphi(x)$, respectively. Note that any $Q(\xi)$ in (2.5) satisfies $|Q(\xi)| = |Q(-\xi)|$ and also

$$Q(\xi + 2k\pi/M) \neq 0 \qquad \text{for some integer } k \text{ with } |k| \leq M/2, \tag{2.6}$$

which follows from (2.4) since $A(\xi)$ is 2π-periodic. Moreover, $Q(\xi)$ in (2.5) is the reduced symbol if and only if

$$Q(2k\pi/M) \neq 0 \qquad \text{for some integer } k \text{ with } 1 \leq k \leq M/2. \tag{2.7}$$

We say that $Q(\xi)$ satisfies *Cohen's condition* if there exists a compact subset F of $\mathbf{R}$, containing a neighborhood of $\xi = 0$ such that the following two conditions hold:

$$\int_F f(\xi)\,d\xi = \int_{[-\pi,\pi]} f(\xi)\,d\xi \tag{2.8a}$$

for any 2π-periodic, non-negative measurable function $f(\xi)$ on $\mathbf{R}$;

$$Q(\xi) \neq 0 \quad \text{for any } \xi \in \bigcup_{j=1}^{\infty} M^{-j}F. \tag{2.8b}$$

It is known [1], [13] that a rank M scaling function $\varphi(x)$ of finite length with symbol $A(\xi)$ satisfying (2.4) is orthogonal if and only if $A(\xi)$ satisfies Cohen's condition. If this is the case, then it follows from (2.3) that $\hat{\varphi} \neq 0$ on F.

It is easy to see that Cohen's condition is satisfied if

$$|Q(\xi)| > 0 \qquad \text{for} \quad \xi \in [0, \pi/M] \tag{2.9}$$

since we can take $F = [-\pi, \pi]$ in Cohen's condition.

The following proposition gives a non-trivial sufficient condition for Cohen's condition, which is a generalization of Corollary 6.3.2 in [1].

Proposition 2.1. *Suppose that there exist an integer m with $1 \leq m \leq M/2$ and a constant c with $2m\pi/\{M(M+1)\} \leq c \leq \pi/M$ such that*

$$Q(\xi) \neq 0 \qquad \text{for} \quad \xi \in [0, c]; \tag{2.10}$$

$$\sum_{k=-m}^{m-1} |Q(\xi + 2k\pi/M)| > 0 \qquad \text{for} \quad \xi \in (c, \pi/M]. \tag{2.11}$$

Then $Q(\xi)$ satisfies Cohen's condition.

Note that, in case M is even, (2.11) is automatically satisfied if $m = M/2$ since $Q(\xi)$ satisfies (2.6) and is 2π-periodic. Thus, we have the following:

Corollary 2.1. *If M is even and $Q(\xi) \neq 0$ for $\xi \in [0, \pi/(M+1)]$, then Q satisfies Cohen's condition.*

On the other hand, in case M is odd, (2.11) is not necessarily satisfied even if $m = (M-1)/2$. See Appendix A for some related examples and the proof of Proposition 2.1.

2.2. Explicit formulae for rank M scaling functions

The following explicit description of a rank M scaling function of degree N whose symbol $A(\xi)$ satisfies (2.4) is indebted to [1], [5]:

First consider the function

$$R_N(\xi) := \sum_{n=0}^{N-1} c_n (1 - \cos\xi)^n,$$

where the coefficients c_n are defined as follows:

When M is even,

$$c_n := \sum_{k_1+k_2+\cdots+k_{M/2}=n} \left\{ \prod_{m=1}^{M/2-1} \binom{2N+k_m-1}{2N-1} \left(1-\cos\frac{2\pi m}{M}\right)^{-k_m} \right\} \times \binom{N+k_{M/2}-1}{N-1} (1-\cos\pi)^{-k_{M/2}}. \tag{2.12a}$$

When M is odd,

$$c_n := \sum_{k_1+k_2+\cdots k_{\frac{M-1}{2}}=n} \left\{ \prod_{m=1}^{\frac{M-1}{2}} \binom{2N+k_m-1}{2N-1} \left(1-\cos\frac{2\pi m}{M}\right)^{-k_m} \right\}. \tag{2.12b}$$

Consider any function $R(\xi)$ of the form $R(\xi) = R_N(\xi) + \tilde{R}(\xi)$ such that

$$\tilde{R}(\xi) := (1-\cos\xi)^N \sum_{\substack{1\le n\le L_0 \\ n\notin M\mathbf{Z}}} \tilde{c}_n \cos n\xi \quad (L_0 \ge 0, L_0 \notin M\mathbf{Z}, \tilde{c}_n \in \mathbf{R}), \tag{2.13}$$

$$R(\xi) \ge 0. \tag{2.14}$$

By the Riesz lemma we can find a trigonometric polynomial $Q(\xi)$ satisfying $|Q(\xi)|^2 = R(\xi)$. Defining a function $A(\xi)$ by formula (2.5), we can obtain a nonzero function $\varphi(x)$ in $L^2(\mathbf{R})$ satisfying (2.3). Any function $\varphi(x)$ constructed above is a rank M scaling function of degree N having finite length L whose symbol is $A(\xi)$ that satisfies (2.4), where L coincides with MN or $MN + L_0 + 1$ according as $\tilde{R}(\xi) \equiv 0$ or $\tilde{R}(\xi) \ne 0$ ($L_0 \ge 1$ and $\tilde{c}_{L_0} \ne 0$). Conversely, it is known [5], [6] that any rank M scaling function $\varphi(x)$ of degree N with finite length whose symbol $A(\xi)$ satisfies (2.4) can be constructed in this way. As noted in the previous subsection, any such scaling function $\varphi(x)$ is orthogonal if and only if its symbol $A(\xi)$ satisfies Cohen's condition. For notational simplicity any rank M orthogonal scaling function $\varphi(x)$ of maximal degree N and length L is denoted by $\varphi_{M,N,L}$. Furthermore, any orthogonal scaling function $\varphi_{M,N,L}$ with $L = MN$, which corresponds to the case where $R(\xi) = R_N(\xi)$, is said to be of *minimal length* and simply denoted by $\varphi_{M,N}$.

2.3. Rank M scaling functions whose reduced symbols have a zero

In this subsection we show the existence of a family of orthogonal rank M scaling functions $\varphi_{M,N,L}$ whose reduced symbols have a zero in $(0,\pi]$. It will be treated in the numerical experiments in §6. We follow the method given in the previous subsection. Let $r(x)$ be a polynomial of the form

$$r(x) = r_N(x) + \tilde{r}(x); \tag{2.15a}$$

$$r_N(x) := \sum_{n=0}^{N-1} c_n(1-x)^n, \quad \tilde{r}(x) := (1-x)^N \sum_{\substack{1\le n\le L_0 \\ n\notin M\mathbf{Z}}} \tilde{c}_n \gamma_n(x), \tag{2.15b}$$

where c_n are given in (2.12), $\tilde{c}_n \in \mathbf{R}$ are arbitrary, and $\gamma_n(x)$ denote the polynomials satisfying $\cos n\xi = \gamma_n(\cos\xi)$. Suppose $r(x) \ge 0$ for any $x \in [-1,1]$. Then we can construct a scaling function φ whose symbol $A(\xi)$ is given by (2.5) with $Q(\xi)$ being a trigonometric polynomial satisfying $|Q(\xi)|^2 = r(\cos\xi)$. Suppose that $Q(\xi)$ has a unique zero ξ_1 in $(0,\pi]$. Note that, in view of (2.15), $Q(\xi)$ cannot have a zero at $\xi_1 = \pi/2$ when $M = 2$. Furthermore, if either $M = 2$, $\xi_1 = \pi$ or $M = 3$, $\xi_1 = 2\pi/3$, then Q is not the reduced symbol of φ since (2.7) fails. While, in case $M \geq 4$, Q becomes the reduced symbol of φ. Moreover, if $0 < \xi_1 < \pi$, then it must hold that $r(\cos\xi_1) = r'(\cos\xi_1) = 0$ since $r(\cos\xi) \geq 0$. In this subsection we restrict ourselves to the case where $\pi/2 \leq \xi_1 \leq \pi$. Thus all the scaling functions obtained below are orthogonal since they satisfy (2.9). The following propositions give a family of such scaling functions $\varphi_{M,N,L}$ with $L = MN + L_0 + 1$ for $L_0 = 1, 2, 3$.

Proposition 2.2. *Let $M = 2$ and $N \geq 1$. For each $\xi_1 \in (3\pi/5, \pi)$, there exists a unique polynomial $r(x)$ of the form (2.15) with $L_0 = 3$ satisfying $r(\cos\xi_1) = r'(\cos\xi_1) = 0$. This polynomial turns out to satisfy $r(\cos\xi) > 0$ for $\xi \in [0,\pi]$ with $\xi \neq \xi_1$. Consequently, there exists an orthogonal scaling function $\varphi_{2,N,2N+4}(x)$ with $r(\cos\xi_1) = r'(\cos\xi_1) = 0$.*

See the remark after the proof of this proposition in Appendix B for the marginal case $\xi_1 = 3\pi/5$.

Proposition 2.3. *Let $M \geq 3$ and $N \geq 1$. There exists a unique polynomial $r(x)$ of the form (2.15) with $L_0 = 1$ satisfying $r(\cos\pi) = 0$. This polynomial turns out to satisfy $r(\cos\xi) > 0$ for $\xi \in [0,\pi)$. Consequently, there exists an orthogonal scaling function $\varphi_{M,N,MN+2}$ with $r(\cos\pi) = 0$.*

Proposition 2.4. *Let $M \geq 3$ and $N \geq 1$. For each $\xi_1 \in [\pi/2, \pi]$ there exists a unique polynomial $r(x)$ of the form (2.15) with $L_0 = 2$ such that $r(\cos\xi_1) = r'(\cos\xi_1) = 0$. This polynomial turns out to satisfy $r(\cos\xi) > 0$ for $\xi \in [0,\pi]$ with $\xi \neq \xi_1$. Consequently, for each $\xi_1 \in [\pi/2, \pi]$, there exists an orthogonal scaling function $\varphi_{M,N,MN+3}(x)$ with $r(\cos\xi_1) = r'(\cos\xi_1) = 0$ except for the case where $M = 3$ and $\xi_1 = 2\pi/3$.*

Suppose $M = 3$ and $\xi_1 = 2\pi/3$. Then it turns out that $r(x) = r_{N+1}(x)$, which generates a scaling function $\varphi_{3,N+1}(x)$ of minimal length.

The proof of these results is given in Appendix B.

3. TRANSFER OPERATORS T_q

In the study of the smoothness of scaling functions through their s_p-exponents, we need to estimate the integral of functions of the form $q(x)q(Mx)\cdots q(M^{j-1}x)f(M^jx)$, where q and f are 2π-periodic, even functions. This function can be obtained from f by the j-times iteration of a suitable operator U_q. The transfer operator T_q is naturally introduced as the adjoint operator of U_q. It turns out that the s_p-exponents of scaling functions are usually represented in terms of the spectral radius of the operator T_q. In this section we study some properties of this operator. In [4, Section 3] basic properties of T_q are studied mainly in the case where q is a polynomial of $\cos\xi$ (for instance, $q = |Q|^2$). In order to estimate the s_p-exponent of scaling function φ, we need to treat the function $q = |Q|^p$ $(p > 0)$, which is Hölder continuous but no more a polynomial of $\cos\xi$ in general. The main result of this section shows that if, at least, q is positive and Hölder continuous, then the spectral radius of T_q coincides with the positive maximum eigenvalue of T_q. This supplements [4, Section 3] and is of interest by itself. Some other related problem are also studied.

Let κ be the piecewise-linear transformation of $[0,\pi]$ onto itself determined by the relation: $\cos\kappa(x) = \cos Mx \quad (0 \le x \le \pi)$, namely, it is given by

$$\kappa(x) := \left|Mx - 2\lceil \tfrac{i}{2} \rceil \pi\right| \quad \text{if} \quad i\pi \le Mx \le (i+1)\pi \ (i = 0, 1, \dots, M-1), \tag{3.1}$$

where $\lceil x \rceil$ denotes the smallest integer not less than x. For each i, let $\theta_i : [0,\pi] \to [\frac{i}{M}\pi, \frac{i+1}{M}\pi]$ be the inverse map of the restriction of κ to $[\frac{i}{M}\pi, \frac{i+1}{M}\pi]$, that is

$$\theta_i(x) := \frac{2\lceil \frac{i}{2} \rceil \pi + (-1)^i x}{M} \qquad (i = 0, 1, \dots, M-1). \tag{3.2}$$

For any measurable functions q and f, we define $T_q(f)$ and $U_q(f)$ by

$$T_q(f)(x) := \sum_{i=0}^{M-1} q(\theta_i(x)) f(\theta_i(x)) \qquad \text{and} \qquad U_q(f)(x) := q(x) f(\kappa(x)).$$

The following three lemmas are immediate from these definitions:

Lemma 3.1. *If q and f are even functions on $\mathbf{R}$, then it holds that*

$$T_q(f)(x) = \sum_{k=\lceil M/2 \rceil - M}^{\lceil M/2 \rceil - 1} q\left(\frac{x + 2k\pi}{M}\right) f\left(\frac{x + 2k\pi}{M}\right) \qquad (x \in [0,\pi]).$$

If, in addition, q and f are 2π-periodic, then it holds that

$$T_q(f)(x) = \sum_{k=0}^{M-1} q\left(\frac{x + 2k\pi}{M}\right) f\left(\frac{x + 2k\pi}{M}\right) \qquad (x \in [0,\pi]).$$

Lemma 3.2. *If q and f are 2π-periodic, even functions on $\mathbf{R}$, then it holds that*

$$U_q^j(f)(x) = q(x) q(Mx) \cdots q(M^{j-1}x) f(M^j x) \quad (x \in [0,\pi];\ j = 1, 2, \dots).$$

Lemma 3.3. *If q, f and g are bounded measurable functions on $[0,\pi]$, then it holds that*

$$\int_0^\pi T_q^j(f) g \, dx = M^j \int_0^\pi f U_q^j(g) \, dx \quad (j = 1, 2, \dots).$$

Let $L^\infty([0,\pi])$ denote the space of all real-valued bounded measurable functions on $[0,\pi]$ with norm $\|f\| := \operatorname{ess\,sup}\{|f(x)| : x \in [0,\pi]\}$ and let $\mathcal{C}([0,\pi])$ denote its subspace consisting of all continuous functions. Consider two cones $K := \{f \in L^\infty([0,\pi]) : f \geq 0 \text{ a.e.}\}$, $K^0 := \{f \in L^\infty([0,\pi]) : \operatorname{ess\,inf}_{[0,\pi]} f > 0\}$ and their restrictions $\widetilde{K} := K \cap \mathcal{C}([0,\pi])$, $\widetilde{K}^0 := K^0 \cap \mathcal{C}([0,\pi])$. For $q \in K$, the operator $T_q : L^\infty([0,\pi]) \to L^\infty([0,\pi])$ is called the *transfer operator* associated with q. This operator is bounded linear and positive in the sense that $T_q(K) \subset K$. The spectral radius of T_q is defined by $\rho(T_q) := \lim_{n\to\infty} \|T_q^n\|^{1/n}$. It is easy to see that $\rho(T_q) = \lim_{n\to\infty} \|T_q^n(f)\|^{1/n}$ for any $f \in K^0$. Hence, if $r \leq q$ a.e. $(r, q \in K)$, then $\rho(T_r) \leq \rho(T_q) \leq M\|q\|$ since $\|T_r^n\| = \|T_r^n(1)\| \leq \|T_q^n\| = \|T_q^n(1)\| \leq (M\|q\|)^n$.

If $q \in \widetilde{K}$, then T_q has $\mathcal{C}([0,\pi])$ as its invariant subspace. Note that the spectral radius of the restriction $\widetilde{T}_q$ of T_q to $\mathcal{C}([0,\pi])$ is the same as $\rho(T_q)$ since $\|\widetilde{T}_q^n\| = \|T_q^n\| = \sup_{x\in[0,\pi]} \widetilde{T}_q^n(1)(x)$. For simplicity, we also denote $\widetilde{T}_q$ by T_q in the following. It is known [4, Lemma 3.2] that if $q \in \widetilde{K}^0$, then for any nonzero $f \in \widetilde{K}$ there exists an integer $k \geq 1$ such that $T_q^k(f) \in \widetilde{K}^0$.

The following theorem gives the basic spectral properties of T_q:

Theorem 3.1. *Suppose $q \in \widetilde{K}^0$ is Hölder continuous and $f \in \widetilde{K}^0$. Let $f_n := T_q^n(f)$ and $g_n := f_n/\|f_n\|$ $(n \geq 1)$. Then*

(i) *There exists a unique $g \in \widetilde{K}$ satisfying $\|g\| = 1$ and $T_q(g) = \lambda g$ for some constant $\lambda > 0$. Furthermore, it turns out that $g \in \widetilde{K}^0$ and $\lambda = \rho(T_q)$;*

(ii) *g_n uniformly converges to g;*

(iii) *$f_{n+1}/f_n \to \rho(T_q)$ uniformly;*

(iv) $\min_{[0,\pi]} (f_{n+1}/f_n) \nearrow \rho(T_q)$ *and* $\max_{[0,\pi]} (f_{n+1}/f_n) \searrow \rho(T_q)$.

In the proof of Theorem 3.1 we use the notion of modulus of continuity. For $f \in \mathcal{C}([0,\pi])$, the *modulus of continuity* $\omega(f,\delta)$ $(\delta > 0)$ of f is defined by $\omega(f,\delta) := \sup_{|x-y|\leq\delta} |f(x) - f(y)|$. Note that $0 \leq \omega(f,\delta) < \infty$ and $\omega(f,\delta) \searrow 0$ $(\delta \to 0)$ since f is uniformly continuous.

Proof of Theorem 3.1. Since $\log q$ is Hölder continuous, we have $\omega(\log q, \delta) \leq C\,\delta^\alpha$ $(\delta > 0)$ for some $C > 0$ and $\alpha \in (0,1]$. If $n \geq 1$, $\delta > 0$ and $|x - y| \leq \delta$, then

$$\begin{aligned} f_{n+1}(y) &= \sum_{i=0}^{M-1} q(\theta_i(y))\, f_n(\theta_i(y)) \\ &\leq \sum_{i=0}^{M-1} \Big[e^{\omega(\log q,\delta/M)} q(\theta_i(x))\Big] \Big[e^{\omega(\log f_n,\delta/M)}\, f_n(\theta_i(x))\Big] \\ &= f_{n+1}(x) e^{\omega(\log q,\delta/M)+\omega(\log f_n,\delta/M)}, \end{aligned}$$

and hence, $\omega(\log f_{n+1}, \delta) \leq \omega(\log q, \delta/M) + \omega(\log f_n, \delta/M)$. It follows that for each $n \geq 1$

$$\begin{aligned} \omega(\log g_n, \delta) = \omega(\log f_n, \delta) &\leq \sum_{i=1}^{n} \omega(\log q, \delta/M^i) + \omega(\log f, \delta/M^n) \\ &\leq \sum_{i=1}^{n} C\,(\delta/M^i)^\alpha + \omega(\log f, \delta/M^n) \leq \eta(\delta), \end{aligned} \tag{3.3}$$

where $\eta(\delta) := \dfrac{C}{M^\alpha - 1} \cdot \delta^\alpha + \omega(\log f, \delta) \searrow 0 \quad (\delta \to 0)$. Therefore, $\{g_n\}_n$ is equicontinuous and $e^{-\eta(\pi)} \le g_n \le 1$ $(n \ge 1)$, which follows from

$$\frac{1}{g_n(x)} = \frac{\|g_n\|}{g_n(x)} \le \sup_{y,z\in[0,\pi]} \frac{g_n(z)}{g_n(y)} = e^{\omega(\log g_n,\pi)} \le e^{\eta(\pi)} \quad (x \in [0,\pi]).$$

By the Ascoli-Arzelà theorem $\{g_n\}_n$ is relatively compact in $\mathcal{C}([0,\pi])$. Based on this fact, all the statements can be established by the standard theory of positive operators. We give the proof, however, for convenience of the reader.

Define $\lambda(f) := \max\bigl(T_q(f)/f\bigr)$ for any $f \in \widetilde{K}^0$. Note that $\lambda(\cdot)$ is continuous on $\widetilde{K}^0$ and that $\lambda(T_q(f)) = \max\bigl(T_q(\{T_q(f)/f\}f)/T_q(f)\bigr) \le \lambda(f)$ and $\lambda(cf) = \lambda(f)$ for any $c > 0$ and $f \in \widetilde{K}^0$. Take any limit point g of $\{g_n\}_n$ and a subsequence $\{g_{n_k}\}_k$ such that $g_{n_k} \to g$. Then, we have $\|g\| = 1$. Noting that $g \in \widetilde{K}^0$ since $g_n \ge e^{-\eta(\pi)}$, we have $\inf_{n\in\mathbf{N}} \lambda(f_n) = \lim_{n\to\infty} \lambda(f_n) = \lim_{k\to\infty} \lambda(f_{n_k}) = \lim_{k\to\infty} \lambda(g_{n_k}) = \lambda(g)$. Now, suppose that $T_q(g) \ne \lambda(g)g$. Then, since $\lambda(g)g - T_q(g) \in \widetilde{K} \setminus \{0\}$, we have $\lambda(g)T_q^m(g) - T_q^{m+1}(g) \in \widetilde{K}^0$ for some $m \in \mathbf{N}$ and hence $\lambda(g) > \lambda(T_q^m(g))$. On the other hand, we have $\lambda(T_q^m(g)) = \lim_{k\to\infty} \lambda(T_q^m(g_{n_k})) = \lim_{k\to\infty} \lambda(f_{m+n_k}) = \inf_{n\in\mathbf{N}} \lambda(f_n) = \lambda(g)$, which is a contradiction. Thus, we have $T_q(g) = \lambda(g)g$. To prove the uniqueness in (i), take any $h \in \widetilde{K}$ satisfying $\|h\| = 1$ and $T_q(h) = \lambda h$ for some constant $\lambda > 0$. Since $T_q^n(h) = \lambda^n h$ $(n \in \mathbf{N})$ and since $T_q^n(h) \in \widetilde{K}^0$ for some n, we have $h \in \widetilde{K}^0$ and hence $\rho(T_q) = \lim_{n\to\infty} \|T_q^n(h)\|^{1/n} = \lambda$. Thus, we have $\lambda(g) = \rho(T_q)$ as well. Suppose $h \ne g$. Then, since $\|g/h\|h - g \in \widetilde{K} \setminus \{0\}$, we have $\|g/h\|T_q^l(h) - T_q^l(g) \in \widetilde{K}^0$ for some $l \in \mathbf{N}$, which leads to contradiction $\|g/h\| > \|T_q^l(g)/T_q^l(h)\| = \|g/h\|$. Hence, we have (i) and also (ii) by the uniqueness of the limit point of $\{g_n\}_n$. Hence, we have $f_{n+1}/f_n = T_q(f_n)/f_n = T_q(g_n)/g_n \to T_q(g)/g = \rho(T_q)$, proving (iii) and (iv) except for the monotonicity, which has already been proved for $\max(f_{n+1}/f_n) = \lambda(f_n)$ and can be proved similarly for $\min(f_{n+1}/f_n)$. ■

In the following propositions we show the continuity of the spectral radius $\rho(T_q)$ and of its normalized eigenfunction in q.

Proposition 3.1 (Continuity of spectral radius). *If $p, q \in K^0$ and $\alpha \le p, q \le \beta$ for some positive constants α and β, then*

$$\frac{\rho(T_q)}{\rho(T_p)} \le \left\|\frac{q}{p}\right\| \le 1 + \frac{\|p-q\|}{\alpha} \quad \text{and} \quad |\rho(T_p) - \rho(T_q)| \le M\,\frac{\beta}{\alpha}\,\|p-q\|.$$

Proof. Since $T_q^j(1) \le \left\|\dfrac{q}{p}\right\|^j T_p^j(1)$ a.e. $(j \ge 1)$, we have

$$\rho(T_q) = \lim_{j\to\infty} \|T_q^j(1)\|^{1/j} \le \left\|\frac{q}{p}\right\| \rho(T_p).$$

The rest follows from this and $\rho(T_q) \le M\|q\| \le M\beta$. ■

Suppose $0 < \alpha \le 1$. Any family $\{f_k\}_k$ in $\mathcal{C}([0,\pi])$ is said to be *equi-α-Hölder continuous*, if $\omega(f_k, \delta) \le C\delta^\alpha$ $(\delta > 0)$ for some constant $C > 0$ independent of k.

PROPOSITION 3.2 (Convergence of eigenfunctions). *Suppose* $q_k \in \widetilde{K}^0$ $(1 \le k \le \infty)$ *are equi-α-Hölder continuous, and* $g_k \in \widetilde{K}^0$, $\|g_k\| = 1$, $T_{q_k}(g_k) = \rho(T_{q_k})g_k$ $(1 \le k \le \infty)$. *If* $q_k \to q_\infty$ *uniformly, then* $g_k \to g_\infty$ *uniformly.*

Proof. Since $q_k \to q_\infty$, it follows that $\log q_k$ $(1 \le k \le \infty)$ are α-Hölder continuous and $\omega(\log q_k, \delta) \le C\delta^\alpha$ $(\delta > 0, 1 \le k \le \infty)$ for some $C > 0$. Let $g_{k,n} := T^n_{q_k}(1)/\|T^n_{q_k}(1)\|$ $(n \ge 1,\ 1 \le k \le \infty)$. It follows from (3.3) that

$$\omega(\log g_{k,n}, \delta) \le \frac{C}{M^\alpha - 1} \cdot \delta^\alpha \quad (\delta > 0, n \ge 1, 1 \le k \le \infty).$$

This means that $\mathcal{G} := \{g_{k,n}\}_{k,n}$ is equicontinuous, and by the Ascoli-Arzelà theorem, $\mathcal{G}$ is relatively compact in $\mathcal{C}([0,\pi])$. Since $g_{k,n} \to g_k$ $(n \to \infty)$ $(1 \le k \le \infty)$ by Theorem 3.1 (ii), it follows that $\{g_k\}_k$ is included in the closure of $\mathcal{G}$. Suppose $g_k \not\to g_\infty$ by contraries. Then there exists a subsequence $\{g_{k_i}\}_i$ converging to some $h \in \widetilde{K}$ with $h \ne g_\infty$. Note that $\|h\| = \lim \|g_{k_i}\| = 1$. It follows from the assumptions on g_{k_i} $(i \ge 1)$ and Proposition 3.1 that $T_{q_\infty}(h) = \rho(T_{q_\infty})h$. Hence, it follows from Theorem 3.1 (i) that $h = g_\infty$, which leads to contradiction. ■

4. THE s_p-EXPONENTS OF RANK M SCALING FUNCTIONS

In this section we are concerned with the s_p-exponents $(p > 0)$ of scaling functions. We retain the notation in §§2 and 3. Let $\varphi(x)$ be a rank M scaling function of maximal degree N with finite length whose symbol $A(\xi)$ satisfies (2.4). Without loss of generality, we can assume that $\hat{\varphi}(0) = 1$. Let $Q(\xi)$ be the reduced symbol of $\varphi(x)$ and let $q(\xi) := |Q(\xi)|^p$. Then the function $q(\xi)$ is a 2π-periodic, even, Hölder continuous function with $q(0) = 1$. In the case of minimal length, $q(\xi)$ is a positive smooth function which is strictly monotone increasing on $[0,\pi]$.

The following is the main theorem of this section, which gives upper and lower bounds for $s_p(\varphi)$ in terms of the transfer operator T_q associated with q. It follows from (2.6) that $T_q(f) \in K^0$ for any $f \in K^0$. Thus, for any $f \in K^0$, we can define $\mu(f) > 0$ and $\lambda(f) > 0$ by

$$\mu(f) := \operatorname*{ess\,inf}_{[0,\pi]}(T_q(f)/f), \qquad \lambda(f) := \operatorname*{ess\,sup}_{[0,\pi]}(T_q(f)/f). \tag{4.1}$$

Note that $\mu(f) \le \rho(T_q) \le \lambda(f)$ $(f \in K^0)$.

THEOREM 4.1. *It holds that*

(i) $s_p(\varphi) \ge N - \dfrac{1}{p}\log_M \rho(T_q)$;

(ii) $s_p(\varphi) \le N - \dfrac{1}{p}\sup\{\log_M \mu(f) : f \in K^0\} \le N$ *if* Q *satisfies Cohen's condition.*

The proof will be given later. See §2.1 for some sufficient conditions for Cohen's condition.

When $q > 0$ on $[0,\pi]$, we obtain an exact formula for the s_p-exponent.

Corollary 4.1. *If $Q(\xi)$ has no zero in $[0,\pi]$, then* $s_p(\varphi) = N - \dfrac{1}{p}\log_M \rho(T_q)$.

This follows from Theorem 4.1 since $\mu(g) = \rho(T_q)$ for the function g in Theorem 3.1 (i).

In the rest of this section we use functions $\Phi(\xi) := \prod_{j=1}^{\infty} Q(\xi/M^j)$ and $h(\xi) := |2\sin(\xi/2)|^N$, and relations (see (2.3) and (2.5)):

$$\hat{\varphi}(\xi) = \left(\prod_{j=1}^{\infty} \frac{1-e^{i\xi/M^{j-1}}}{M(1-e^{i\xi/M^j})}\right)^N \Phi(\xi) = \left(e^{i\,(\xi/2)}\,\frac{\sin(\xi/2)}{\xi/2}\right)^N \Phi(\xi), \tag{4.2}$$

$$|\hat{\varphi}(\xi)| = |\hat{\varphi}(-\xi)|, \qquad |\hat{\varphi}(\xi)| = |\xi|^{-N} h(\xi)|\Phi(\xi)|, \tag{4.3}$$

$$|\Phi(\xi)|^p = q(\xi/M)|\Phi(\xi/M)|^p. \tag{4.4}$$

The following lemma is needed in the proof of Theorem 4.1.

Lemma 4.1. *It holds that*

(i) $\rho(T_q) \geq 1$;

(ii) $\mu(f) \geq 1$ *for some* $f \in \widetilde{K}^0$ *if* Q *satisfies Cohen's condition.*

Proof. Assertion (i) follows from $\|T_q^n\| \geq T_q^n(1)(0) \geq q(0)^n = 1$ ($n \in \mathbf{N}$). To prove (ii) take the compact set F in Cohen's condition (2.8). Then there exists an integer $r \geq 1$ such that $[-\pi,\pi] \subset \bigcup_{k=-Mr}^{Mr}(F - 2k\pi)$. Consider a continuous function

$$f(\xi) := \sum_{k=-Mr+\lceil M/2\rceil - M}^{Mr+\lceil M/2\rceil - 1} |\Phi(\xi + 2k\pi)|^p.$$

Then $f > 0$ on $[-\pi,\pi]$ since $|\Phi| > 0$ on F. Let $g_k(\xi) := |\Phi(\xi + 2k\pi)|^p$. It follows from (4.4) and Lemma 3.1 that

$$\begin{aligned} f(\xi) &= \sum_{n=-r}^{r} \sum_{k=\lceil M/2\rceil - M}^{\lceil M/2\rceil - 1} q\left(\frac{\xi + 2(Mn+k)\pi}{M}\right)\left|\Phi\left(\frac{\xi + 2(Mn+k)\pi}{M}\right)\right|^p \\ &= \sum_{k=\lceil M/2\rceil - M}^{\lceil M/2\rceil - 1} q\left(\frac{\xi + 2k\pi}{M}\right) \sum_{n=-r}^{r} g_n\left(\frac{\xi + 2k\pi}{M}\right) \\ &= T_q\left(\sum_{n=-r}^{r} g_n\right)(\xi) \leq T_q(f)(\xi) \qquad (\xi \in [0,\pi]). \end{aligned}$$

Hence, we have $\mu(f) \geq 1$. ■

Proof of Theorem 4.1. Let $R := r\pi$ ($r \in \mathbf{N}$) and let $s_0 := s - N$. Using (4.3), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} |(1+|\xi|)^s \hat{\varphi}(\xi)|^p d\xi < \infty \qquad (4.5) \\ &\iff \int_{|\xi|>R} |\xi|^{sp} |\hat{\varphi}(\xi)|^p d\xi = \int_{|\xi|>R} |\xi|^{s_0 p} h(\xi)^p |\Phi(\xi)|^p d\xi < \infty \\ &\iff \sum_{k=1}^{\infty} M^{s_0 pk} (I_k - I_{k-1}) < \infty, \end{aligned}$$

where $I_k := \displaystyle\int_{|\xi|\le M^k R} h(\xi)^p |\Phi(\xi)|^p \, d\xi \quad (k = 1, 2, \ldots)$. When $s_0 < 0$, any one of these conditions is equivalent to the following condition

$$\sum_{k=1}^{\infty} M^{s_0 pk} I_k < \infty. \qquad (4.6)$$

Indeed, this equivalence follows from

$$\sum_{k=1}^{n} M^{s_0 pk} (I_k - I_{k-1}) = (1 - M^{s_0 p}) \sum_{k=1}^{n-1} M^{s_0 pk} I_k + M^{s_0 pn} I_n - M^{s_0 p} I_0.$$

Note that

$$\begin{aligned} I_k &= M^k \int_{-R}^{R} h(M^k \xi)^p |\Phi(M^k \xi)|^p \, d\xi \\ &= M^k \int_{-R}^{R} h(M^k \xi)^p \left(\prod_{j=0}^{k-1} q(M^j \xi) \right) |\Phi(\xi)|^p \, d\xi \end{aligned}$$

and consider

$$\begin{aligned} J_k &:= M^k \int_{-\pi}^{\pi} h(M^k \xi)^p \prod_{j=0}^{k-1} q(M^j \xi) \, d\xi \\ &= 2M^k \int_0^{\pi} U_q^k(h^p) \, d\xi = 2 \int_0^{\pi} h^p T_q^k(1) \, d\xi. \end{aligned}$$

To prove (i) let $m_1 := \max_{[-R,R]} |\Phi|^p$. Since $h(\xi) \le 2^N$, we have

$$M^{s_0 pk} I_k \le m_1 r M^{s_0 pk} J_k \le m_1 r 2^{Np+1} \pi M^{s_0 pk} \|T_q^k(1)\|.$$

Suppose $s < N - \dfrac{1}{p} \log_M \rho(T_q)$. Then, since $M^{s_0 p} \|T_q^k(1)\|^{1/k} \to M^{s_0 p} \rho(T_q) < 1$ ($k \to \infty$), we have (4.6). Since $s_0 < 0$ by Lemma 4.1 (i), we have (4.5) and hence $s \le s_p(\varphi)$.

To prove (ii) let F be the compact set in Cohen's condition and let $m_0 := \min_F |\Phi|^p > 0$. We take $R = r\pi$ so that $F \subset [-R, R]$. Since q and h are 2π-periodic, it follows that

$$I_k \ge m_0 M^k \int_F h(M^k \xi)^p \prod_{j=0}^{k-1} q(M^j \xi) \, d\xi = m_0 J_k. \qquad (4.7)$$

Take any $f \in K^0$. Then since $T_q^k(f) \geq \mu(f)^k f$ $(k \geq 1)$, it follows that for each $k \geq 1$

$$M^{s_0pk} J_k \geq M^{s_0pk} \frac{2}{\|f\|} \int_0^\pi h^p T_q^k(f)\, d\xi \geq (M^{s_0p}\mu(f))^k \frac{2}{\|f\|} \int_0^\pi h^p f\, d\xi. \qquad (4.8)$$

Suppose $s < \min\{s_p(\varphi), N\}$. Since (4.5) holds and $s_0 < 0$, it follows from (4.6), (4.7) and (4.8) that $s < N - \dfrac{1}{p}\log_M \mu(f)$ and hence $\min\{s_p(\varphi), N\} \leq N - \dfrac{1}{p}\log_M \mu(f)$. By taking a special f in Lemma 4.1, we have $\min\{s_p(\varphi), N\} = s_p(\varphi)$, proving (ii). ■

Now we give the proof of (1.1).

PROPOSITION 4.1. *If $f \in L^2(\mathbf{R})$ has compact support, then inequalities* (1.1) *hold true.*

Proof. Suppose that $0 < p \leq 1$. Since f has compact support, we have $f \in L^1(\mathbf{R})$ and hence $\hat{f} \in L^\infty(\mathbf{R})$. Taking any $s < s_p(f)$, we have

$$\int_{-\infty}^{\infty} |(1+|\xi|)^{sp} \hat{f}(\xi)|\, d\xi \leq \sup_{\mathbf{R}} |\hat{f}(\xi)|^{1-p} \int_{-\infty}^{\infty} |(1+|\xi|)^s \hat{f}(\xi)|^p\, d\xi < \infty$$

and hence $sp \leq s_1(f)$. Letting $s \nearrow s_p(f)$, we have $ps_p(f) \leq s_1(f)$. Suppose, in turn, that $p > 1$ and let $p' > 0$ be such that $1/p + 1/p' = 1$. Take any $s < s_p(f)$. If $s < t < s_p(f)$, then we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} |(1+|\xi|)^{s-1/p'} \hat{f}(\xi)|\, d\xi \\
&\leq \Big(\int_{-\infty}^{\infty} |(1+|\xi|)^t \hat{f}(\xi)|^p\, d\xi\Big)^{1/p} \Big(\int_{-\infty}^{\infty} (1+|\xi|)^{p'(s-t)-1}\, d\xi\Big)^{1/p'} < \infty
\end{aligned}$$

by using the Hölder inequality. This proves that $s - 1/p' \leq s_1(f)$. Letting $s \nearrow s_p(f)$, we have $s_p(f) - (p-1)/p = s_p(f) - 1/p' \leq s_1(f)$. Combining these, we have the first inequality in (1.1).

To prove the second inequality, we can assume with no loss of generality that $s_1(f) > 0$ since $\alpha(f) \geq 0$. Take any $s < s_1(f)$. Then, we can choose $n \in \mathbf{N} \cup \{0\}$ and $\sigma \in (0,1)$ such that $s \leq n + \sigma < s_1(f)$, which implies that $f \in \mathcal{C}^n$. Noting that $|1 - e^{i\theta}| \leq |\theta| \wedge 2 \leq 2(|\theta| \wedge 1)^\sigma \leq 2|\theta|^\sigma$ $(\theta \in \mathbf{R})$, we have

$$\begin{aligned}
|f^{(n)}(x) - f^{(n)}(y)| &= \Big|\frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\xi)^n \hat{f}(\xi) e^{-i\xi x}(1 - e^{i\xi(x-y)})\, d\xi\Big| \\
&\leq \frac{1}{\pi} \int_{-\infty}^{\infty} |\xi|^n |\hat{f}(\xi)| |\xi(x-y)|^\sigma\, d\xi \\
&\leq \frac{|x-y|^\sigma}{\pi} \int_{-\infty}^{\infty} (1+|\xi|)^{n+\sigma} |\hat{f}(\xi)|\, d\xi < \infty \qquad (x, y \in \mathbf{R}),
\end{aligned}$$

which means that $f \in \mathcal{C}^{n+\sigma}$ and hence $s \leq n + \sigma \leq \alpha(f)$. Letting $s \nearrow s_1(f)$, we have $s_1(f) \leq \alpha(f)$.

Finally, we prove the third inequality. Since $\hat{f} \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}) \subset L^r(\mathbf{R})$ for any $r \geq 2$, we have $s_r(f) \geq 0$ $(r \geq 2)$. Thus, we can assume that $\alpha(f) > 0$ with no loss of generality. Take any $\alpha = n + \sigma < \alpha(f)$ with $n \in \mathbf{N} \cup \{0\}$, $0 < \sigma < 1$ and let $\tau \in (\sigma, 1)$ be such that $\alpha < n + \tau < \alpha(f)$. Note that $f \in \mathcal{C}^{n+\tau}$ and hence $f^{(n)} \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ since $f^{(n)} \in \mathcal{C}^\tau$ has also compact support. Define

$$f^{(\alpha)}(x) := \frac{1}{c} \int_{-\infty}^{\infty} \frac{f^{(n)}(x+y) - f^{(n)}(x)}{|y|^{1+\sigma}}\, dy, \tag{4.9}$$

where $c := 2\displaystyle\int_0^\infty \frac{1-\cos y}{y^{1+\sigma}}\, dy$. To prove that $f^{(\alpha)} \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, let $f_1^{(\alpha)}(x)$ be defined by the right hand side of (4.9) with the integral being restricted to $|y| \leq 1$ and let $f_2^{(\alpha)} := f^{(\alpha)} - f_1^{(\alpha)}$. Since $f^{(n)}(x) = 0$ $(|x| \geq R)$ for some $R > 0$, we have $f_1^{(\alpha)}(x) = 0$ $(|x| > R+1)$ and

$$|f_1^{(\alpha)}(x)| \leq \frac{1}{c} \int_{|y| \leq 1} \frac{dy}{|y|^{1+\sigma-\tau}} \sup_{z \neq w} \frac{|f^{(n)}(z) - f^{(n)}(w)|}{|z-w|^\tau} < \infty,$$

proving that $f_1^{(\alpha)} \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Denoting by $\|\cdot\|_k$ the norms of $L^k(\mathbf{R})$ for $k = 1, 2$, we have

$$\begin{aligned} \|f_2^{(\alpha)}\|_k &\leq \frac{1}{c} \int_{|y|>1} \frac{\|f^{(n)}(\cdot + y) - f^{(n)}\|_k}{|y|^{1+\sigma}}\, dy \\ &\leq \frac{2}{c} \int_{|y|>1} \frac{dy}{|y|^{1+\sigma}} \|f^{(n)}\|_k < \infty. \end{aligned}$$

Thus, we have $\widehat{f^{(\alpha)}} \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}) \subset L^r(\mathbf{R})$ $(r \geq 2)$. Since $|\widehat{f^{(\alpha)}}(\xi)| = |\xi|^\alpha |\hat{f}(\xi)|$, we have, for any $r \geq 2$,

$$\int_{-\infty}^{\infty} |(1+|\xi|)^\alpha \hat{f}(\xi)|^r\, d\xi \leq (2^{\alpha-1} \vee 1)^r \int_{-\infty}^{\infty} (|\hat{f}(\xi)| + |\widehat{f^{(\alpha)}}(\xi)|)^r\, d\xi < \infty,$$

proving that $\alpha \leq s_r(f)$. Letting $\alpha \nearrow \alpha(f)$, we have $\alpha(f) \leq s_r(f)$ $(r \geq 2)$. ■

The following upper bound on the Sobolev exponent of scaling function $\varphi(x)$ is given in [6].

PROPOSITION 4.2 ([6]). *If $0 < \xi_0 < 2\pi$, $M\xi_0 \equiv \pm\xi_0$ mod 2π and $\Phi(\xi_0) \neq 0$, then $s_2(\varphi) \leq N - \log_M |Q(\xi_0)|$. In particular, $s_2(\varphi) \leq N - \log_M |Q(\xi_c)|$, where $\xi_c := M\pi/(M+1)$ or π according as M is even or odd.*

Since $\alpha(\varphi) \leq s_2(\varphi)$, this statement also gives an upper bound on $\alpha(\varphi)$. However, we can derive this upper bound on $\alpha(\varphi)$ directly from the Riemann-Lebesgue theorem.

PROPOSITION 4.3. *If $0 < \xi_0 < 2\pi$, $M\xi_0 \equiv \pm\xi_0$ mod 2π and $\Phi(\xi_0) \neq 0$, then $\alpha(\varphi) \leq N - \log_M |Q(\xi_0)|$.*

Proof. In case $\alpha(\varphi) > 0$, take any $\alpha = n+\sigma < \alpha(\varphi)$ with $n \in \mathbf{N}\cup\{0\}$, $0 < \sigma < 1$. Since $\varphi \in \mathcal{C}^\beta$ for some β with $\alpha < \beta < \alpha(\varphi)$ and since φ has compact support, we can define an integrable function $\varphi^{(\alpha)}$ by (4.9) with $f = \varphi$. In case $\alpha(\varphi) = 0$, take $\alpha = 0$ so that $\varphi^{(\alpha)} := \varphi$ is also integrable. It follows from the Riemann-Lebesgue theorem that $\widehat{\varphi^{(\alpha)}}(\xi) \to 0$ $(\xi \to \infty)$. On the other hand, it follows from (4.3) that

$$|\widehat{\varphi^{(\alpha)}}(\xi)| = |\xi|^\alpha|\hat{\varphi}(\xi)| = |\xi|^{\alpha-N}h(\xi)|\Phi(\xi)|.$$

Now let $\xi_k := M^k\xi_0$ $(k \geq 1)$ and note that

$$\begin{aligned}
\sin(\xi_k/2) &= \sin(M^k\xi_0/2) = \pm\sin(\xi_0/2) \neq 0, \\
|\Phi(\xi_k)| &= \prod_{j=1}^{\infty}|Q(M^{k-j}\xi_0)| = |Q(\xi_0)|^k|\Phi(\xi_0)| = M^{k\gamma}|\Phi(\xi_0)| \neq 0,
\end{aligned}$$

where $\gamma := \log_M |Q(\xi_0)|$. Thus, we have

$$\begin{aligned}
|\widehat{\varphi^{(\alpha)}}(\xi_k)| &= |\xi_k|^{\alpha-N}h(\xi_k)|\Phi(\xi_k)| \\
&= |M^k\xi_0|^{\alpha-N}h(\xi_0)M^{k\gamma}|\Phi(\xi_0)| \\
&= M^{k(\alpha-N+\gamma)}|\xi_0|^{\alpha-N}h(\xi_0)|\Phi(\xi_0)|.
\end{aligned}$$

Since $\widehat{\varphi^{(\alpha)}}(\xi_k) \to 0$ $(k \to \infty)$, we have $\alpha - N + \gamma < 0$ and hence $\alpha(\varphi) \leq N - \gamma$. ■

5. APPROXIMATION BY STEP FUNCTIONS

In this section we are concerned with comparison and finite-dimensional approximation for $\rho(T_q)$ and for $s_p(\varphi)$ by step functions. For any integer $m \geq 1$, let $S_m([0,\pi])$ denote the subspace of $L^\infty([0,\pi])$ consisting of all step functions that are constant on each interval $(\frac{i-1}{m}\pi, \frac{i}{m}\pi)$ $(i = 1,\ldots,m)$. For each i, let $\chi_i^{(m)} \in S_m([0,\pi])$ denote the indicator function of the interval $(\frac{i-1}{m}\pi, \frac{i}{m}\pi)$ $(i = 1,2,\ldots,m)$. Then $\{\chi_i^{(m)}\}_{i=1}^m$ forms a basis of $S_m([0,\pi])$.

The following lemma can be proved by a direct calculation except for the last assertion, which is a consequence of the classical Perron-Frobenius theory for a positive matrix.

LEMMA 5.1. *Suppose $M \geq 2$, $m \geq 1$ and $q = \sum_{i=1}^{M^2m} q_i\chi_i^{(M^2m)} \in S_{M^2m}([0,\pi])$, where $q_i \geq 0$.*

(i) $T_q(S_{Mm}([0,\pi])) \subset S_{Mm}([0,\pi])$.

(ii) *With respect to the basis $\{\chi_i^{(Mm)}\}$, the linear map $T_q : S_{Mm}([0,\pi]) \to S_{Mm}([0,\pi])$ is represented by the $Mm \times Mm$-matrix $W_q = (V_1,\cdots,V_M)$, where V_j $(j = 1,2,\ldots,M)$ are the $Mm \times m$-matrices given in the following:*

In case j is odd,

$$V_j := \begin{pmatrix} q_{Mm(j-1)+1} & & & & & \\ \vdots & & & & & \\ q_{Mm(j-1)+M} & & & & & \\ & q_{Mm(j-1)+M+1} & & & & \\ & \vdots & & & & \\ & q_{Mm(j-1)+2M} & & & & \\ & & \vdots & & & \\ & & & \vdots & & \\ & & & & \vdots & \\ & & & & & q_{Mm(j-1)+(m-1)M+1} \\ & & & & & \vdots \\ & & & & & q_{Mmj} \end{pmatrix};$$

in case j is even,

$$V_j := \begin{pmatrix} & & & & & q_{Mmj} \\ & & & & & \vdots \\ & & & & & q_{Mm(j-1)+(m-1)M+1} \\ & & & & \vdots & \\ & & & \vdots & & \\ & & \vdots & & & \\ & q_{Mm(j-1)+2M} & & & & \\ & \vdots & & & & \\ & q_{Mm(j-1)+M+1} & & & & \\ q_{Mm(j-1)+M} & & & & & \\ \vdots & & & & & \\ q_{Mm(j-1)+1} & & & & & \end{pmatrix}.$$

(iii) *The spectral radius $\rho(T_q)$ of T_q coincides with the positive maximum eigenvalue of the positive matrix W_q.*

The matrix W_q in the above lemma in case $M = 3$ and $m = 4$, for example, is of the form

$$W_q = \left(\begin{array}{cccc|cccc|cccc} q_1 & & & & & & & q_{24} & q_{25} & & & \\ q_2 & & & & & & & q_{23} & q_{26} & & & \\ q_3 & & & & & & & q_{22} & q_{27} & & & \\ & q_4 & & & & & q_{21} & & & q_{28} & & \\ & q_5 & & & & & q_{20} & & & q_{29} & & \\ & q_6 & & & & & q_{19} & & & q_{30} & & \\ & & q_7 & & & q_{18} & & & & & q_{31} & \\ & & q_8 & & & q_{17} & & & & & q_{32} & \\ & & q_9 & & & q_{16} & & & & & q_{33} & \\ & & & q_{10} & q_{15} & & & & & & & q_{34} \\ & & & q_{11} & q_{14} & & & & & & & q_{35} \\ & & & q_{12} & q_{13} & & & & & & & q_{36} \end{array}\right).$$

In the rest of this section, we apply these observations to estimating $s_p(\varphi)$ and $\rho(T_q)$, where φ is the scaling function investigated in §4 with $Q(\xi)$ being its reduced symbol and $q(\xi) = |Q(\xi)|^p$ $(p > 0)$. For each $m \geq 1$, let q^- and q^+, respectively, be the maximal and the minimal elements in $S_{M^2m}([0,\pi])$ satisfying $q^- \leq q \leq q^+$ a.e. Take any $f = \sum_{i=1}^{Mm} v_i \chi_i^{(Mm)} \in S_{Mm}([0,\pi]) \cap K^0$. Then, by the above lemma, we can define $f_n^{\pm} := T_{q^{\pm}}^n(f) \in S_{Mm}([0,\pi])$ $(n = 1, 2, \ldots)$. Since $Q(\xi)$ satisfies (2.6), if m is sufficiently large, then we have $f_n^{\pm} \in K^0$ so that we can define μ_n^-, μ_n^+, λ_n^-, λ_n^+ by

$$\mu_n^{\pm} := \operatorname{ess\,inf}(f_{n+1}^{\pm}/f_n^{\pm}), \quad \lambda_n^{\pm} := \operatorname{ess\,sup}(f_{n+1}^{\pm}/f_n^{\pm}) \quad (n = 1, 2, \ldots). \tag{5.1}$$

In view of Lemma 5.1, these quantities can be obtained by using the matrices $W_{q^{\pm}}$: Letting ${}^t(v_{n,1}^{\pm}, \ldots, v_{n,Mm}^{\pm}) = W_{q^{\pm}}^n {}^t(v_1, \ldots, v_{Mm})$, we have

$$\mu_n^{\pm} = \min_{1 \leq i \leq Mm} (v_{n+1,i}^{\pm}/v_{n,i}^{\pm}), \quad \lambda_n^{\pm} = \max_{1 \leq i \leq Mm} (v_{n+1,i}^{\pm}/v_{n,i}^{\pm}) \quad (n = 1, 2, \ldots) \tag{5.2}$$

since $f_n^{\pm} = \sum_{i=1}^{Mm} v_{n,i}^{\pm} \chi_i^{(Mm)}$. Note that

$$\mu_n^{\pm} \leq \mu_{n+1}^{\pm} \leq \rho(T_{q^{\pm}}) \leq \lambda_{n+1}^{\pm} \leq \lambda_n^{\pm} \qquad (n = 1, 2, \ldots). \tag{5.3}$$

Further, if $q^{\pm} \in K^0$, then we can apply the classical Perron-Frobenius theory (see [7] for example) to $T_{q^{\pm}}$ and a suitable modification of Theorem 3.1 is valid for $T_{q^{\pm}}$. In particular, we have

$$\mu_n^{\pm} \nearrow \rho(T_{q^{\pm}}) \text{ and } \lambda_n^{\pm} \searrow \rho(T_{q^{\pm}}) \quad \text{as } n \to \infty. \tag{5.4}$$

Based on these relations we obtain the following two theorems which are useful in numerical experiments:

THEOREM 5.1. *Let φ be a rank M scaling function of maximal degree N with finite length whose symbol $A(\xi)$ satisfies* (2.4). *Let μ_n^- and λ_n^+ $(n = 1, 2, \ldots)$ be defined by* (5.1) *or* (5.2) *for sufficiently large m. Then, it holds that*

(i) $s_p(\varphi) \geq N - \frac{1}{p} \log_M \rho(T_{q^+}) = N - \frac{1}{p} \inf_n \log_M \lambda_n^+ = N - \frac{1}{p} \lim_{n\to\infty} \log_M \lambda_n^+$*;*

(ii) $s_p(\varphi) \leq N - \frac{1}{p} \sup_n \log_M \mu_n^- = N - \frac{1}{p} \lim_{n\to\infty} \log_M \mu_n^-$ *if Q satisfies Cohen's condition.*

Proof. Assertion (i) follows from Theorem 4.1 (i), (5.3) and (5.4) for λ_n^+ since $\rho(T_q) \leq \rho(T_{q^+})$ and $q^+ \in K^0$. Assertion (ii) follows from Theorem 4.1 (ii) and $\mu_n^- = \operatorname{ess\,inf} \left(T_{q^-}(f_n^-)/f_n^-\right) \leq \operatorname{ess\,inf} \left(T_q(f_n^-)/f_n^-\right)$ $(n = 1, 2, \ldots)$. ∎

THEOREM 5.2. *Let φ and μ_n^-, λ_n^+ $(n = 1, 2, \ldots)$ be the same as in Theorem 5.1. Suppose that $Q(\xi)$ has no zero in $[0, \pi]$. Then it holds that*

$$\rho(T_q) = \sup_{m,n} \mu_n^- = \inf_{m,n} \lambda_n^+ = \lim_{m,n\to\infty} \mu_n^- = \lim_{m,n\to\infty} \lambda_n^+, \tag{5.5}$$

$$s_p(\varphi) = N - \frac{1}{p} \lim_{m,n\to\infty} \log_M \mu_n^- = N - \frac{1}{p} \lim_{m,n\to\infty} \log_M \lambda_n^+. \tag{5.6}$$

Proof. Since $q^- \to q$ and $q^+ \to q$ uniformly as $m \to \infty$, it follows from Proposition 3.1 that $\rho(T_q) = \lim_{m\to\infty} \rho(T_{q^-}) = \sup_m \rho(T_{q^-})$ and $\rho(T_q) = \lim_{m\to\infty} \rho(T_{q^+}) = \inf_m \rho(T_{q^+})$. Moreover, since $q^-, q^+ \in K^0$ for any m, (5.5) follows from (5.4). Combining (5.5) with Theorem 5.1, we have (5.6) ■

6. NUMERICAL RESULTS

This section includes numerical results for the s_1-exponent $s_1(\varphi_{M,N,L})$. From theoretical view point, we could use Theorem 4.1 directly. However, it still remains to evaluate the global infimum and supremum in (4.1) numerically. This problem can be solved by an approximation method if appropriate rigorous bounds for errors are available. The iterative method in §5 gives an answer to this problem. Thus the following numerical results are mainly based upon Theorems 5.1 and 5.2.

6.1. Minimal-length case

To the minimal-length scaling function $\varphi_{M,N}$ we can apply Theorem 5.2 since $q = |Q| > 0$. Recall the bounds for the Hölder exponents $\alpha(\varphi_{M,N})$ known so far (see (1.1) and Proposition 4.2):

$$s_2(\varphi_{M,N}) - \frac{1}{2} \le s_1(\varphi_{M,N}) \le \alpha(\varphi_{M,N}) \le s_2(\varphi_{M,N}) \le N - \log_M |Q(\xi_c)|. \quad (6.1)$$

Tables 1 and 2 include some of our numerical results for $s_1(\varphi_{M,N})$ when $M = 2$ and 3, respectively. For comparison, the numerical results for $s_2(\varphi_{M,N}) - 1/2$ and $N - \log_M |Q(\xi_c)|$ are also given in the tables, where most of the numerical results for $s_2(\varphi_{M,N})$ are taken from [6, Table 3.1].

6.2. Non-minimal-length case

The smoothness of a non-minimal-length scaling function $\varphi_{M,N,L}$ can be remarkably improved if the polynomial $r(x)$ with $|Q(\xi)|^2 = r(\cos\xi)$ has zeros of some orders at periodic or preperiodic points of the transformation κ. This fact was shown in [6] for the Sobolev regularity. Tables 3 and 4 give some of our numerical results for the s_1-exponents of the following scaling functions:

(i) $\varphi_{2,N,2N+4}$ with $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = 2\pi/3, 3\pi/4, 4\pi/5, 5\pi/6$;

(ii) $\varphi_{3,N,3N+2}$ with $r(\cos\pi) = 0$; $\varphi_{3,N,3N+3}$ with $r(\cos\xi) = r'(\cos\xi) = 0$ for $\xi = \pi/2, 3\pi/4, 5\pi/6, \pi$.

Note that Propositions 2.2–2.4 assure the existence of these scaling functions, and they also show that, for each $\varphi_{M,N,L}$ above, the associated polynomial $r(x)$ is uniquely determined by the condition of the type $r(\cos\xi) = r'(\cos\xi) = 0$ or $r(\cos\pi) = 0$. Consequently, the quantities of the type $s_1(\varphi_{M,N,L})$ with $r(\cos\xi) = r'(\cos\xi) = 0$ or $r(\cos\pi) = 0$ in Tables 3 and 4 are well defined since the s_1-exponent of $\varphi_{M,N,L}$ only depends on the associated polynomial $r(x)$. The numerical results in these tables are due to the lower and upper bounds for $s_1(\varphi_{M,N,L})$ given in Theorem 5.1. Despite lack of any exact expression for $s_1(\varphi_{M,N,L})$, in contrast with

TABLE 1

$s_2(\varphi_{2,N}) - 1/2$, $s_1(\varphi_{2,N})$ and $N - \log_2 |Q(2\pi/3)|$

$M = 2$				
N	L	$s_2(\varphi_{2,N}) - 1/2$	$s_1(\varphi_{2,N})$	$N - \log_2 \|Q(2\pi/3)\|$
1	2	0	0	1
2	4	0.5000	0.521	1.339
3	6	0.9150	0.980	1.636
4	8	1.2756	1.392	1.913
5	10	1.5968	1.768	2.177
6	12	1.8884	2.117	2.432
7	14	2.1587	2.442	2.682
8	16	2.4147	2.747	2.927
9	18	2.6617	3.036	3.168
10	20	2.9027	3.310	3.406
11	22	3.1398	3.572	3.641
12	24	3.3740	3.826	3.875
13	26	3.6060	4.072	4.106

TABLE 2

$s_2(\varphi_{3,N}) - 1/2$, $s_1(\varphi_{3,N})$ and $N - \log_3 |Q(\pi)|$

$M = 3$				
N	L	$s_2(\varphi_{3,N}) - 1/2$	$s_1(\varphi_{3,N})$	$N - \log_3 \|Q(\pi)\|$
1	3	0	0	1
2	6	0.4087	0.443	1.160
3	9	0.6599	0.779	1.254
4	12	0.7950	1.031	1.321
5	15	0.8665	1.211	1.373
6	18	0.9133	1.331	1.415
7	21	0.9499	1.410	1.450
8	24	0.9809	1.462	1.481
9	27	1.0081	1.499	1.508
10	30	1.0323	1.528	1.532
11	33	1.0542	1.552	1.554
12	36	1.0741	1.573	1.574
13	39	1.0925	1.592	1.592
77	231	1.4999	1.999	1.999
78	234	1.5016	2.002	2.002

Theorem 5.2, we have succeeded in determining their values down to three or four places of decimals by considerable amount of computer works. This fact suggests the exact formula $s_1(\varphi) = N - \log_M \rho(T_q)$ to hold under much weaker conditions on Q than that in Corollary 4.1.

TABLE 3

s_1-exponents of rank $M = 2$ scaling functions of non-minimal length.

		$s_1(\varphi_{2,N,2N+4})$ with $r(\cos\xi) = r'(\cos\xi) = 0$			
N	L	$\xi = 2\pi/3$	$\xi = 3\pi/4$	$\xi = 4\pi/5$	$\xi = 5\pi/6$
1	6	−0.057	0.342	0.488	0.579
2	8	0.482	0.981	1.153	1.257
3	10	0.990	1.556	1.736	1.837
4	12	1.468	2.073	2.240	2.326
5	14	1.923	2.540	2.669	2.728
6	16	2.358	2.961	3.030	3.055
7	18	2.778	3.342	3.343	3.332
8	20	3.186	3.690	3.623	3.584
9	22	3.584	4.011	3.886	3.824
10	24	3.975	4.310	4.136	4.058
11	26	4.358	4.592	4.379	4.290
12	28	4.735	4.859	4.616	4.520
13	30	5.107	5.114	4.850	4.748

TABLE 4

s_1-exponents of rank $M = 3$ scaling functions of non-minimal length.

$s_1(\varphi_{3,N,3N+2})$ with $r(\cos\xi) = 0$			$s_1(\varphi_{3,N,3N+3})$ with $r(\cos\xi) = r'(\cos\xi) = 0$					
N	L	$\xi = \pi$	N	L	$\xi = \pi/2$	$\xi = 3\pi/4$	$\xi = 5\pi/6$	$\xi = \pi$
1	5	0.176	1	6	−0.095	0.463	0.416	0.351
2	8	0.708	2	9	0.125	0.996	1.014	0.925
3	11	1.158	3	12	0.262	1.399	1.529	1.416
4	14	1.550	4	15	0.350	1.702	1.983	1.847
5	17	1.903	5	18	0.409	1.912	2.380	2.236
6	20	2.227	6	21	0.451	2.041	2.686	2.595
7	23	2.530	7	24	0.482	2.115	2.864	2.933
8	26	2.817	8	27	0.508	2.161	2.959	3.257
9	29	3.089	9	30	0.529	2.193	3.016	3.570
10	32	3.348	10	33	0.548	2.219	3.053	3.874
11	35	3.596	11	36	0.566	2.241	3.081	4.171
12	38	3.832	12	39	0.581	2.260	3.103	4.460
13	41	4.057	13	42	0.596	2.278	3.122	4.743

APPENDIX A

On Cohen's condition

First we give the proof of Proposition 2.1, which is a straightforward generalization of that of [1, Corollary 6.3.2].

Proof of Proposition 2.1. Let $\xi_1, \ldots, \xi_n$ be the zeros of $Q(\xi)$ contained in the interval $(c, \pi/M]$. It follows from (2.11) that for each i there exists an integer k_i with $-m \leq k_i \leq m-1$ such that $|Q(\xi_i + 2k_i\pi/M)| > 0$. Let $I_i := (\xi_i - \varepsilon, \xi_i + \varepsilon) \cap (c, \pi/M]$ for each $i = 1, 2, \ldots, n$, where $\varepsilon > 0$ is chosen to be sufficiently small so that I_i $(i = 1, 2, \ldots, n)$ do not intersect each other and that $\inf_{\xi \in I_i} |Q(\xi + 2k_i\pi/M)| > 0$ $(i = 1, 2, \ldots, n)$. Set $F := F_1 \cup F_2$, where

$$F_1 := [-\pi, \pi] \setminus \{G \cup (-G)\} \qquad \text{with } G := \bigcup_{i=1}^{n} MI_i,$$

$$F_2 := H \cup (-H) \qquad \text{with } H := \bigcup_{i=1}^{n} \overline{(MI_i + 2k_i\pi)}.$$

It suffices to show that $|Q(\xi/M^j)| > 0$ for any $\xi \in F$ $(j = 1, 2, \ldots)$. Note that $2\pi \leq 2m\pi \leq M(M+1)c$ and $Mc \leq \pi$. Suppose $\xi \in F_1$. We can assume, by symmetry, that $\xi \in [0, \pi] \setminus G$. Then since $\xi/M \in [0, \pi/M] \setminus \bigcup_{i=1}^{n} I_i$, we have $|Q(\xi/M)| > 0$. If $j \geq 2$, then $|Q(\xi/M^j)| > 0$ since $0 \leq \xi \leq \pi \leq M(M+1)c/2 \leq M^2c$. Suppose $\xi \in F_2$. Then we can assume that $\xi \in H$ by symmetry. Then since $\xi/M \in \bigcup_{i=1}^{n} \overline{(I_i + 2k_i\pi/M)}$, we have $|Q(\xi/M)| > 0$. Moreover, since $\xi \leq \pi + 2(m-1)\pi = 2m\pi - \pi \leq M(M+1)c - Mc = M^2c$ and $\xi > Mc - 2m\pi \geq Mc - M(M+1)c = -M^2c$, we have $|\xi/M^2| \leq c$ and hence $|Q(\xi/M^j)| > 0$ if $j \geq 2$. ■

In the rest of this section we give some examples of scaling functions related to Cohen's condition. All scaling functions below can be constructed as in §2.2. The first example shows that the right end point π/M of the interval in (2.9) cannot be decreased in case M is odd, while, in case M is even, it can be reduced to $\pi/(M+1)$ as shown in Corollary 2.1.

Example A.1. A rank 3 scaling function $\varphi(x)$ with symbol $A(\xi) := (1 + e^{2i\xi} + e^{4i\xi})/3$ is of degree $N = 1$ and has length $L = 5$. Its reduced symbol $Q(\xi)$ and the polynomial $r(x)$ satisfying $|Q(\xi)|^2 = r(\cos\xi)$ are as follows:

$$Q(\xi) := e^{2i\xi} - e^{i\xi} + 1,$$
$$r(x) := 4(x - 1/2)^2 = 1 - 4(1-x)x.$$

Hence, $Q(\xi)$ has a unique zero in $[0, \pi]$ at $\xi = \pi/3$. It is obvious that $\varphi(x)$ is not orthogonal since $\varphi(x) = \frac{1}{2} 1_{[0,2]}(x)$, where $1_{[0,2]}$ denotes the indicator function of interval $[0, 2]$. Consequently, Cohen's condition is not satisfied for this example.

Note that point π/M is a pre-fixed point of transformation κ in §3 if M is odd; a similar role was played by the point $\pi/(M+1) = \pi/3$ when $M = 2$ as mentioned in [1, pp. 187–188]. This fact and the above example suggest that the point $\pi/3$

is critical for rank 3 scaling function in the sense that, in some neighborhood of $\pi/3$, only $\pi/3$ cannot be a zero of its reduced symbol $Q(\xi)$ that satisfies Cohen's condition. In fact, Proposition 2.1 with $M = 3$, $m = 1$ and $c = \pi/6$ assures that the reduced symbol $Q(\xi)$ having zeros in $(\pi/6, \pi/3)$ satisfies Cohen's condition if only $Q \neq 0$ on $[0, \pi/6] \cup [\pi/3, \pi/2)$. The following is such an example:

Example A.2. Consider a rank 3 scaling function $\varphi(x)$ of degree $N = 1$ with length $L = 6$ whose symbol $A(\xi)$, reduced symbol $Q(\xi)$ and the polynomial $r(x)$ satisfying $|Q(\xi)|^2 = r(\cos\xi)$ are given by

$$\begin{aligned} A(\xi) &:= \frac{1}{6}\left\{(1+\sqrt{2}) + e^{2i\xi} + (1-\sqrt{2})e^{3i\xi} + 2e^{4i\xi} + e^{5i\xi}\right\}, \\ Q(\xi) &:= \frac{1}{2}(e^{i\xi} + 1 + \sqrt{2})(e^{2i\xi} - \sqrt{2}e^{i\xi} + 1), \\ r(x) &:= 2(1+\sqrt{2})(x - 1/\sqrt{2})^2(x+\sqrt{2}) = 1 - (1+\sqrt{2})(1-x)\{2x + (2x^2-1)\}. \end{aligned}$$

Since $r(\cos\pi/4) = r'(\cos\pi/4) = 0$ and $r(x) > 0$ for $x \in [-1, 1]$ with $x \neq \cos\pi/4$, it follows from Proposition 2.1 with $m = 1$ and $c = \pi/6$ that Q satisfies Cohen's condition.

The following example shows that, in contrast with the case $M = 3$, $Q(\xi)$ satisfying Cohen's condition can have a zero at π/M when $M = 5$.

Example A.3. Consider the following function $Q(\xi)$ and $r(x)$:

$$\begin{aligned} Q(\xi) &:= \frac{1}{2(1-x_0)(1+a)^2}(e^{2i\xi} - 2x_0 e^{i\xi} + 1)(e^{i\xi} + a)^2, \quad \text{where} \\ x_0 &:= \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}, \\ a &:= 3x_0 - 1 + \sqrt{9x_0^2 - 6x_0} = \frac{1}{4}\left(-1 + 3\sqrt{5} + \sqrt{30 - 6\sqrt{5}}\right), \\ r(x) &:= \frac{8(3+\sqrt{5})}{9}\left(x - \frac{1+\sqrt{5}}{4}\right)^2\left(x - \frac{1-3\sqrt{5}}{4}\right)^2 \\ &= 1 - \frac{6+2\sqrt{5}}{9}(1-x)\left\{(2+\sqrt{5})x + 2\sqrt{5}(2x^2-1) + (4x^3-3x)\right\}. \end{aligned}$$

There exists a rank 5 scaling function $\varphi(x)$ of degree $N = 1$ with length $L = 9$ having $Q(\xi)$ as its reduced symbol. Polynomial $r(x)$ satisfies $|Q(\xi)|^2 = r(\cos\xi)$. Since $r(\cos\pi/5) = r'(\cos\pi/5) = 0$ and $r(x) > 0$ for $x \in [-1, 1]$ with $x \neq \cos\pi/5$, it follows from Proposition 2.1 with $m = 2$ and $c = 2\pi/15$ that Q satisfies Cohen's condition.

APPENDIX B

Proof of Propositions 2.2–2.4

In this section we prove Propositions 2.2–2.4. To this end, we consider functions $\sigma(x)$, $\sigma_N(x)$ and $\tilde{\sigma}(x)$ defined for $x \in [-1, 1)$ by

$$\sigma(x) := r(x)(1-x)^{-N} = \sigma_N(x) + \tilde{\sigma}(x);$$
$$\sigma_N(x) := r_N(x)(1-x)^{-N} = \sum_{n=0}^{N-1} c_n(1-x)^{n-N}, \quad \tilde{\sigma}(x) := \sum_{\substack{1\le n\le L_0 \\ n\notin M\mathbf{Z}}} \tilde{c}_n \gamma_n(x),$$

where $r(x)$, $r_N(x)$, c_n, $\tilde{c}_n$ and $\gamma_n(x)$ are given in (2.15). Note that $\sigma_N(x)$, $\sigma_N'(x)$, $\sigma_N''(x) > 0$ since $c_n > 0$ $(n = 0, 1, 2, \ldots)$.

Proof of Proposition 2.2. Suppose $\xi_1 \in (3\pi/5, \pi)$ so that $x_1 := \cos\xi_1 \in (-1, x_0)$, where $x_0 := \cos 3\pi/5 = (1-\sqrt{5})/4 < 0$. Since $L_0 = 3$, we can assume that $\tilde{\sigma}(x) = \tilde{c}_1 x + \tilde{c}_3(4x^3 - 3x)$. Solving equations $\sigma(x_1) = \sigma'(x_1) = 0$, we have

$$\tilde{c}_1 = \frac{(12x_1^2 - 3)\sigma_N(x_1) - (4x_1^3 - 3x_1)\sigma_N'(x_1)}{-8x_1^3}, \quad \tilde{c}_3 = \frac{x_1\sigma_N'(x_1) - \sigma_N(x_1)}{-8x_1^3} < 0,$$

which determines $\sigma(x)$ and $r(x)$ uniquely. Since $r(x)$ is a polynomial of degree $N+3$ satisfying $r(x_1) = r'(x_1) = 0$, it can be written in the following two forms:

$$r(x) = \left(1 - \frac{1-x}{1-x_1}\right)^2 \cdot \sum_{n=0}^{N+1} b_n(1-x)^n = \sum_{n=0}^{N+3} d_n(1-x)^n, \tag{B.1}$$

where $d_n = c_n$ $(n = 0, 1, \ldots, N-1)$ and $d_N, \ldots, d_{N+3}$ are determined by $\tilde{c}_1$ and $\tilde{c}_3$; for example, $d_{N+3} = -4\tilde{c}_3$ and $d_{N+2} = 12\tilde{c}_3$. We follow the convention that $(1-x)^0 = 1$. Observing

$$\sum_{n=0}^{N+1} b_n(1-x)^n = \left(1 - \frac{1-x}{1-x_1}\right)^{-2} r(x)$$
$$= \sum_{n=0}^{\infty}(n+1)\left(\frac{1-x}{1-x_1}\right)^n \cdot \sum_{k=0}^{N+3} d_k(1-x)^k$$

when $|1-x| < |1-x_1|$, we have

$$b_n = \sum_{k=0}^{n}(n-k+1)c_k(1-x_1)^{k-n} > 0 \qquad (n = 0, 1, \ldots, N-1).$$

In particular, we have

$$b_{N-1} = \sum_{k=0}^{N-1}(N-k)c_k(1-x_1)^{k-N+1} = (1-x_1)^2\,\sigma_N'(x_1).$$

Comparing the coefficients of $(1-x)^n$ with $n = N+3$ and $N+2$ in (B.1), we have

$$\frac{b_{N+1}}{(1-x_1)^2} = d_{N+3} = -4\tilde{c}_3,$$
$$\frac{b_N}{(1-x_1)^2} = \frac{2}{1-x_1} b_{N+1} + d_{N+2} = 4(1+2x_1)\tilde{c}_3.$$

Consider the following quadratic polynomial of z:

$$P_N(z) := \frac{1}{(1-x_1)^2}(b_{N+1}z^2 + b_N z + b_{N-1}) = -4\tilde{c}_3 z^2 + 4(1+2x_1)\tilde{c}_3 z + \sigma_N'(x_1).$$

Noting that $-4\tilde{c}_3 > 0$, we can minimize this as follows:

$$\min_z P_N(z) = P_N\left(\frac{1+2x_1}{2}\right) = \frac{\{x_1\sigma_N'(x_1) - \sigma_N(x_1)\}(1+2x_1)^2 - 8x_1^3\sigma_N'(x_1)}{-8x_1^3}.$$

Using $(1-x_1)\sigma_N'(x_1) = \sum_{n=0}^{N-1} c_n(N-n)(1-x_1)^{n-N} \geq \sigma_N(x_1)$, we have

$$\min_z P_N(z) \geq \frac{\sigma_N'(x_1)}{-8x_1^3}(4x_1^2 - 2x_1 - 1) > 0$$

since $x_1 < x_0$ and since x_0 is the minimum real root of $4x^2 - 2x - 1$. Thus we have

$$r(x) = \left(\frac{x-x_1}{1-x_1}\right)^2 \left\{\sum_{n=0}^{N-2} b_n(1-x)^n + (1-x)^{N-1}(1-x_1)^2 P_N(1-x)\right\} > 0.$$

for $x \in [-1,1]$ with $x \neq x_1$. ■

Remark. Consider the case $\xi_1 = 3\pi/5$. As seen from the above proof, there also exists a unique $r(x)$. Moreover, if $N \geq 2$, then $r(x) > 0$ for $x \in [-1,1]$ with $x \neq x_1$ since $\sum_{n=0}^{N-2} b_n(1-x)^n > 0$. While, in case $N = 1$, we have $r(x) = 16(x-\cos 3\pi/5)^2(x-\cos\pi/5)^2$, which turns out to generate a non-orthogonal scaling function $\frac{1}{5}1_{[0,5]}(x)$ with symbol $A(\xi) = (e^{5i\xi}+1)/2$.

Proof of Proposition 2.3. Since $L_0 = 1$, we can assume that $\tilde{\sigma}(x) = \tilde{c}_1 x$. Solving $\sigma(-1) = 0$, we have $\tilde{c}_1 = \sigma_N(-1) > 0$, which uniquely determines $r(x)$. Suppose that $x > -1$. Since $\sigma'(x) = \sigma_N'(x) + \tilde{c}_1 > 0$, we have $\sigma(x) > \sigma(-1) = 0$ and hence $r(x) > 0$. ■

Proof of Proposition 2.4. Since $L_0 = 2$, we can assume that $\tilde{\sigma}(x) = \tilde{c}_1 x + \tilde{c}_2(2x^2 - 1)$. Let $x_1 := \cos\xi_1 \leq 0$. Solving $\sigma(x_1) = \sigma'(x_1) = 0$, we have

$$\tilde{c}_1 = \frac{-4x_1\sigma_N(x_1) + (2x_1^2-1)\sigma_N'(x_1)}{2x_1^2+1}, \quad \tilde{c}_2 = \frac{-x_1\sigma_N'(x_1) + \sigma_N(x_1)}{2x_1^2+1} > 0,$$

which determines $r(x)$ uniquely. Suppose that $-1 \leq x \leq 1$ and $x \neq x_1$. Since $\sigma''(x) = \sigma_N''(x) + 4\tilde{c}_2 > 0$, we have $\sigma(x) > 0$ and, hence, $r(x) > 0$. ■

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