LETTER

Diffraction Amplitudes from Periodic Neumann Surface: Low Grazing Limit of Incidence

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SUMMARY This paper deals with the diffraction of TM plane wave by a perfectly conductive periodic surface. Applying the Rayleigh hypothesis, a linear equation system determining the diffraction amplitudes is derived. The linear equation is formally solved by Cramer's formula. It is then found that, when the angle of incidence becomes a low grazing limit, the amplitude of the specular reflection becomes −1 and any other diffraction amplitudes vanish for any perfectly conductive periodic surfaces with small roughness and gentle slope.

key words: *wave diffraction, TM wave, periodic grating, perfectly conductive surface, diffraction amplitudes*

1. Introduction

When a plane wave is incident on a flat surface, the reflection takes place. For any angle of incidence, the reflection coefficient is −1 in case of the Dirichlet surface but becomes 1 in case of Neumann surface. In case of the Neumann rough surface, however, several approximate theories show that the reflection coefficient changes its sign and becomes −1 at a low grazing limit of incidence with $\theta_i \to 0$ or π (See Fig. 1). Here, by the Dirichlet and Neumann surfaces, we mean the perfectly conductive surface for TE and TM waves, respectively. Such a behavior of the reflection coefficient at a low grazing limit was discussed for randomly rough surfaces [1]–[6] and periodic rough surfaces [7]. This problem has received much interest, because it is closely related with the wave propagation along the rough sea and sea echo observation by a ground based radar [8].

However, these works were all restricted to a slightly rough case. On the other hand, we found numerically that such a behavior is true in case of a sinusoidal grating with moderate surface roughness [9] and a periodic array of rectangular grooves with deep groove depth. These facts give us an expectation such that the reflection coefficient must be −1 at a low grazing limit of incidence for a periodic or random Neumann surface, regardless of the surface roughness and surface shape. Such an expectation seems to be new but is difficult to prove it in general case. By use of the Rayleigh hypothesis, however, we give a simple mathematical description to show that such an expectation is true for any periodic Neumann surfaces with small roughness and gentle slope.

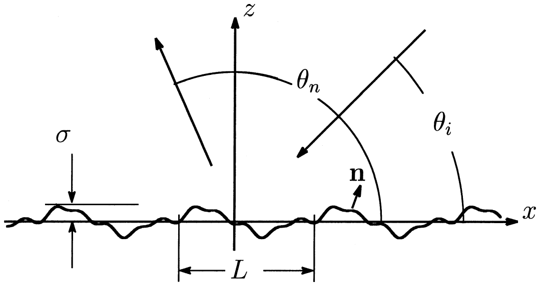

Fig. 1 Diffraction of a plane wave from a periodic surface. L is the period, $\sigma = \max[f(x)]$ is the highest excursion of the surface. The angle of incidence θ_i and the nth order diffraction angle θ_n are measured from the positive x axis.

2. Formulation and TM Case

Let us consider the diffraction of a TM plane wave by a periodic surface shown in Fig. 1. We write the surface profile as

$$z = f(x) = f(x + L), \quad k_L = 2\pi/L, \tag{1}$$

where L is the period and k_L is the spatial angular frequency of L. In TM case, we denote the y component of the magnetic field by $\psi(x, z)$, which satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\psi(x, z) = 0, \quad z > f(x), \tag{2}$$

in the free space above the surface (1) and the Neumann boundary condition

$$\left.\frac{\partial\psi(x, z)}{\partial n}\right|_{z=f(x)} = 0, \tag{3}$$

where k is wavenumber and n is normal. By $\sigma = \max\{f(x)\}$, we denote the highest excursion of the surface. Due to the periodicity of the surface, the wave field in the region $z > \sigma$ may have the Floquet form,

$$\psi(x, z) = e^{-ipx-i\beta_0 z} + e^{-ipx}\sum_{m=-\infty}^{\infty} A_m e^{-imk_L x+i\beta_m z}, \quad (z > \sigma), \tag{4}$$

where the first term in the right-hand side is the incident plane wave, and the second the diffracted wave. Here, p and β_m are given by

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$$\beta_m = \sqrt{k^2 - (p + mk_L)^2},$$

$$\mathrm{Re}[\beta_m] \geq 0, \quad \mathrm{Im}[\beta_m] \geq 0, \quad (m = 0, \pm 1, \pm 2, \cdots), \tag{5}$$

$$p = k\cos\theta_i, \quad \beta_0 = k\sin\theta_i, \tag{6}$$

where θ_i is the angle of incidence. The A_0 is the reflection coefficient and A_m is the diffraction amplitude of the mth order Floquet mode, which is diffracted into the θ_m direction. Here, θ_m is determined by the grating formula:

$$k\cos\theta_m = -(p + mk_L), \quad (m = 0, \pm 1, \pm 2, \cdots). \tag{7}$$

Let us obtain the diffraction amplitude A_m by use of the Rayleigh hypothesis [10]. Assuming that the series (4) and its normal derivative converge on the surface (1), we substitute (4) into (3) to obtain

$$\begin{aligned} \sqrt{1 + \left(\frac{df}{dx}\right)^2} \frac{\partial\psi}{\partial n}\bigg|_{z=f} &= i\left[p\frac{df}{dx} - \beta_0\right] e^{-ipx - i\beta_0 f(x)} \\ &\quad + i\sum_{m=-\infty}^{\infty}\left[(p + mk_L)\frac{df}{dx} + \beta_m\right] \\ &\quad \times e^{-i(p+mk_L)x + i\beta_m f(x)} A_m = 0. \end{aligned} \tag{8}$$

Multiplying $e^{i(p+lk_L)x}$ to this equation and integrating the result over one period, we obtain a set of equations for the diffraction amplitudes,

$$\sum_{m=-\infty}^{\infty} q_{l,m}A_m = -e_l, \quad (l = 0,\ \pm 1,\ \pm 2,\ \cdots), \tag{9}$$

$$q_{l,m} = \frac{1}{L}\int_{-L/2}^{L/2}\left[\frac{df}{dx}(p + mk_L) + \beta_m\right] e^{i(l-m)k_L x + i\beta_m f(x)} dx, \tag{10}$$

$$e_l = \frac{1}{L}\int_{-L/2}^{L/2}\left[p\frac{df}{dx} - \beta_0\right] e^{ilk_L x - i\beta_0 f(x)} dx. \tag{11}$$

Here, e_l represents the excitation by the incident plane wave.

However, we introduce A'_m the modified diffraction amplitude by the relation

$$A_m = -\delta_{m,0} + A'_m. \tag{12}$$

From (12) and (9), one finds the equation for A'_m as

$$\sum_{m=-\infty}^{\infty} q_{l,m}A'_m = q_{l,0} - e_l = 2\beta_0 r_l, \tag{13}$$

where r_l is given by

$$\begin{aligned} r_l = \frac{1}{L}\int_{-L/2}^{L/2} \bigg\{ &\cos[\beta_0 f(x)] \\ &+ ip\frac{df}{dx}\frac{\sin[\beta_0 f(x)]}{\beta_0}\bigg\} e^{ilk_L x} dx. \end{aligned} \tag{14}$$

If $f(x)$ and df/dx are finite for any x, r_l becomes finite for any l. Thus, the right hand side of (13) vanishes when $\beta_0 = k\sin(\theta_i) \to 0$. This suggests that the modified diffraction amplitude A'_m vanishes in the limit $\beta_0 \to 0$.

Physically, (9) means that the diffraction amplitude A_m is excited by the incident plane wave. On the other hand, (13) implies that the modified diffraction amplitude A'_m is generated by

$$\psi_p(x, z) = e^{-ipx}e^{-i\beta_0 z} - e^{-ipx}e^{i\beta_0 z}, \tag{15}$$

which is the sum of the incident plane wave and the specularly reflected wave with reflection coefficient −1. In fact, we have the normal derivative $\partial\psi_p/\partial n$ on the surface as

$$\sqrt{1 + \left(\frac{df}{dx}\right)^2} \frac{\partial\psi_p}{\partial n}\bigg|_{z=f}$$

$$= -2i\beta_0\left\{\cos[\beta_0 f(x)] + ip\frac{df}{dx}\frac{\sin[\beta_0 f(x)]}{\beta_0}\right\} e^{-ipx} \tag{16}$$

$$= -2i\beta_0 \sum_{l=-\infty}^{\infty} r_l e^{-i(p+lk_L)x}, \tag{17}$$

where r_l is given by (14). Here, (16) means that $\psi_p(x, z)$ satisfies the Neumann condition (3) on the surface when $\beta_0 = k\sin(\theta_i) \to 0$.

Let us consider the solution A'_m. Introducing a truncation number N, we approximate the infinite sum in (13) by a finite sum from $m = -N$ to N. In other words, we assume $A_m = 0$ for $|m| \geq N + 1$. Then, we solve the truncated (13) by Cramer's formula to obtain the modified diffraction amplitude A'_m,

$$A'_m = 2\beta_0\frac{\mathcal{R}_m}{Q}, \quad (m = 0, \pm 1, \pm 2, \cdots, \pm N), \tag{18}$$

where Q and $\mathcal{R}_m$ are the determinants given by

$$\begin{array}{cc} & \begin{array}{ccccc} (-N) & & (m) & & (N) \end{array} \\ Q = [0] & \left(\begin{array}{ccccc} q_{-N,-N} & \cdots & q_{-N,m} & \cdots & q_{-N,N} \\ q_{1-N,-N} & \cdots\cdots & q_{1-N,m} & \cdots & q_{1-N,N} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ q_{0,-N} & \cdots & q_{0,m} & \cdots & q_{0,N} \\ q_{1,-N} & \cdots & q_{1,m} & \cdots & q_{1,N} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ q_{N,-N} & \cdots & q_{N,m} & \cdots & q_{N,N} \end{array}\right), \end{array} \tag{19}$$

$$\begin{array}{cc} & \begin{array}{ccccc} (-N) & & (m) & & (N) \end{array} \\ \mathcal{R}_m = [0] & \left(\begin{array}{ccccc} q_{-N,-N} & \cdots & r_{-N} & \cdots & q_{-N,N} \\ q_{1-N,-N} & \cdots\cdots & r_{1-N} & \cdots & q_{1-N,N} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ q_{0,-N} & \cdots & r_0 & \cdots & q_{0,N} \\ q_{1,-N} & \cdots & r_1 & \cdots & q_{1,N} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ q_{N,-N} & \cdots & r_N & \cdots & q_{N,N} \end{array}\right). \end{array} \tag{20}$$

Here, (m) indicates the mth column and [0] means the 0th row. The determinant $\mathcal{R}_m$ can be obtained from (19) by replacing $q_{l,m}$ in the mth column with r_l.

Substituting the solution (18) into (12), we have

$$A_m = -\delta_{m,0} + 2\beta_0 \frac{\mathcal{R}_m}{Q}. \tag{21}$$

The determinant Q does not vanish in general. But an exception takes place when the surface is flat and $f(x)$ is identically equal to 0. In such a flat case, we obtain $q_{l,m} = \beta_m \delta_{l,m}$ and $r_l = \delta_{l,0}$ from (10) and (14). Then we find $\mathcal{R}_m = 0$ when $m \neq 0$ and $\mathcal{R}_0/Q = 1/\beta_0$. As a result, we have $A_m = \delta_{m,0}$ in the flat case. For a corrugated surface, it holds that $Q \neq 0$ even when $\beta_0 = 0$, and hence we find from (21)

$$\lim_{\beta_0 \to 0} A_m = \begin{cases} -1, & (m = 0) \\ 0, & (m \neq 0) \end{cases}, \tag{22}$$

which is the main result of this paper. This means that the reflection coefficient A_0 becomes -1 and any other diffraction coefficient A_m, $(m \neq 0)$, vanishes at a low grazing limit with $\theta_i \to 0$ or π. This fact was found analytically by a grazing perturbation method in case of slightly rough periodic surface [7]. This fact is also supported by several numerical examples as is described above. We note that (22) holds formally for any truncation number N and for any period L, if N and L are finite.

3. Conclusions

By use of the Rayleigh hypothesis, we give a simple mathematical discussion on the diffraction of TM plane wave by a periodic Neumann surface. We formally find that, at a low grazing limit of incidence, the specular reflection coefficient becomes -1 and any other diffraction amplitudes vanish for any periodic Neumann surface. As is well known, however, the series (4) converges in the region above the highest excursion of the surface but often diverges on the surface (1). Generally speaking, the Rayleigh hypothesis becomes valid when the surface is small in roughness and gentle in slope and when $f(x)$ is analytical function of x [10], [11]. Therefore, we conclude that, at a low grazing limit of incidence, the specular reflection coefficient becomes -1 and any other diffraction amplitudes vanish for any analytical Neumann surface with small roughness and gentle sloping.

As is described above, however, our expectation (22) holds numerically in case of a sinusoidal grating with moderate surface roughness and a periodic array of rectangular grooves with deep groove depth. Therefore, we expect that (22) holds even when the surface has large roughness and edges. Further, the relation (16) suggests that the Rayleigh hypothesis could hold for any periodic Neumann surface at a low grazing limit of incidence. However, mathematical discussions on such cases will be left for future study. Physically, multiple scattering processes that yield (22) must be clarified. However, this problem is also left for future study.

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