DIRECT COMPUTATION OF THE ANALYTIC TORSIONS OF THE LINE BUNDLES OVER $P^n(\mathbf{C})$

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(Received October 15, 2004)

1. Introduction

The analytic torsions of the Hermitian line bundles [1] over the complex projective spaces are calculated in Ling Weng's paper [2] using the arithmetic Riemann-Roch formula. In the paper, Ling Weng did not use the information on the eigenvalues of the $\bar{\partial}$-Laplacians and asked how the eigenvalues may relate to the calculation. We shall show here the representation theoretical calculation of the eignevalues and the direct computation of the analytic torsion. We hope this paper contributes to the better understanding of the nature of the analytic torsion.

2. The $\bar{\partial}$-Laplacian

We first give the general representation theoretical formula for the $\bar{\partial}$-Laplacian on a compact Hermitian symmetric space.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$, K a connected closed subgroup of G with Lie albegra $\mathfrak{k}$. We fix a positive definite invariant bilinear form $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbf{R}$ and denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. We assume that (G,K) is a symmetric pair. We have the relations

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m},\quad [\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k},\quad [\mathfrak{k},\mathfrak{m}]\subset\mathfrak{m},\quad [\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}.$$

We further assume that K has a central element j whose adjoint action $J=Ad(j)$ gives a complex structure on $\mathfrak{m}$. The quotient space $M=G/K$ is a Hermitian symmetric space of the compact type with the Riemannian metric defined by B restricted to $\mathfrak{m}$.

The Lie group G is a principal K-bundle over M, and is endowed with a G-invariant connection whose horizontal space is spanned by the elements of $\mathfrak{m}$, considered as the left invariant vector fields. A $\mathbf{C}$-vector space V is called a K-module when it has a K-action $\chi\colon K\to \mathrm{Aut}_{\mathbf{C}}(V)$. We always fix a K-invariant Hermitian inner product on V. For a K-module (χ,V), we have an associated Hermitian vector bundle $E=G\times_K V$. The space of smooth sections $C^\infty(E)$ is identified with a subspace $C^\infty(G,K;V)$ of the space of smooth V-valued functions $C^\infty(G;V)$ defined by

$$C^\infty(G,K;V)=\left\{ s\in C^\infty(G;V) \;\middle|\; s(gk)=\chi(k^{-1})s(g)\quad g\in G, k\in K\right\}.$$

For example, the complexification of the space $\mathcal{X}(M)$ of smooth vector fields on M is identified with

$C^{\infty}(G,K;\mathfrak{m}^{\mathbf{C}})$, where the complexification $\mathfrak{m}^{\mathbf{C}}$ of $\mathfrak{m}$ is a K-module with respect to the adjoint action. In the sequel, we usually treat the complex-valued objects, and sometimes omit the symbol of the complexification.

Since E has an induced connection, we can define a covariant derivative $\nabla_X s$ for $X \in \mathcal{X}(M)$ and $s \in C^{\infty}(E)$, which is a section of $C^{\infty}(E)$. When we consider X as an $\mathfrak{m}$-valued function and both s and $\nabla_X s$ as V-valued functions on G, we have an identity $(\nabla_X s)(g) = (X(g)s)(g)$ $(g \in G)$, where $X(g) \in \mathfrak{m}$ is considered as a left invariant vector field on G. The covariant differentiation $D: C^{\infty}(E) \to C^{\infty}(E \otimes T^* M)$ corresponds to the map $D: C^{\infty}(G,K;V) \to C^{\infty}(G,K;V \otimes \mathfrak{m}^*)$ defined by

$$(Ds)(g)(X) = (Xs)(g), \qquad s \in C^{\infty}(G,K;V), \quad g \in G, \quad X \in \mathfrak{m}.$$

We denote by $\mathfrak{m}_-$ [resp. $\mathfrak{m}_+$] the anti-holomorphic [resp. holomorphic] part of the complexification of $\mathfrak{m}$. The holomorphic structure of E is given by the $(0,1)$-component $\bar\partial$ of the complex linear extension of D, i.e., $(\bar\partial s)(g)(X) = (Xs)(g)$, $(s \in C^{\infty}(G,K;V),\ g \in G,\ X \in \mathfrak{m}_-)$. For $k \in K$, we have $(\bar\partial s)(gk)(X) = \chi(k^{-1})(\bar\partial s)(g)(Ad(k)X)$, as is expected.

This $\bar\partial$-operator extends to the tensor bundle with the exterior product of the anti-holomorphic cotangent bundle:

$$\bar\partial: C^{\infty}(E \otimes T^{0,q}M) \to C^{\infty}(E \otimes T^{0,q+1}M)$$

is defined by

$$(\bar\partial s)(X_1,\dots,X_{q+1}) = \sum_{i=1}^{q+1}(-1)^{i-1}X_i(s(X_1,\dots,\widehat{X_i},\dots,X_{q+1}))$$

for $X_1,\dots,X_{q+1} \in \mathfrak{m}_-$, where $\widehat{\ }$ means the deletion of the member. We notice that $[X,Y]$ vanishes for $X, Y \in \mathfrak{m}_-$, since $[X,Y] \in \mathfrak{k}^{\mathbf{C}}$ satisfies

$$[X,Y] = Ad(j)[X,Y] = [JX,JY] = [-\sqrt{-1}X, -\sqrt{-1}Y] = -[X,Y].$$

We have $\bar\partial^2 = 0$ and $\bar\partial$ gives a complex over a holomorphic vector bundle E, as is seen by

$$\begin{aligned}
&(\bar\partial(\bar\partial s))(X_1,\dots,X_{q+2})\\
&= \sum_{i=1}^{q+2}(-1)^{i-1}X_i((\bar\partial s)(X_1,\dots,\widehat{X_i},\dots,X_{q+2}))\\
&= \sum_{j<i}(-1)^{i+j}X_i(X_j(s(X_1,\dots,\widehat{X_j},\dots,\widehat{X_i},\dots,X_{q+2})))\\
&\quad - \sum_{i<j}(-1)^{i+j}X_i(X_j(s(X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{q+2})))\\
&= \sum_{j<i}(-1)^{i+j}[X_i,X_j](s(X_1,\dots,\widehat{X_j},\dots,\widehat{X_i},\dots,X_{q+2}))\\
&= 0.
\end{aligned}$$

We fix an orthonormal basis $\{E_a\}$ of $\mathfrak{m}_-$. The complex conjugate $\{\overline{E}_a\}$ forms an orthonormal basis of $\mathfrak{m}_+$, and satisfies $B(\overline{E}_a, E_b) = \delta_{ab}$. The K-invariant Hermitian inner product on $V \otimes \Lambda^q \mathfrak{m}_-^*$ and the Haar measure on G induces a K-invariant Hermitian inner product on $C^{\infty}(E \otimes T^{0,q}M)$. The formal adjoint of the $\bar\partial$-operator to this inner product is given by

$$(\bar\partial^* s)(X_1,\dots,X_q) = -\sum_a \bar{E}_a(s(E_a,X_1,\dots,X_q)).$$

We now define the $\bar\partial$-Laplacian $\Box$ on $C^\infty(E\otimes T^{0,q}M)$ by $\Box s = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)s$.

$$\begin{aligned}
&(\Box s)(X_1,\dots,X_q)\\
&\quad = ((\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)s)(X_1,\dots,X_q)\\
&\quad = \sum_{i=1}^{q}(-1)^{q-1}X_i((\bar\partial^* s)(X_1,\dots,\widehat{X_i},\dots,X_q))\\
&\qquad - \sum_a \bar{E}_a((\bar\partial s)(E_a,X_1,\dots,X_q))\\
&\quad = -\sum_a\sum_{i=1}^{q}(-1)^{i-1}X_i(\bar{E}_a(s(E_a,X_1,\dots,\widehat{X_i},\dots,X_q)))\\
&\qquad - \sum_a \bar{E}_a(E_a(s(X_1,\dots,X_q)))\\
&\qquad + \sum_a\sum_{i=1}^{q}(-1)^{i-1}\bar{E}_a(X_i(s(E_a,X_1,\dots,\widehat{X_i},\dots,X_q)))\\
&\quad = -\sum_a \frac{1}{2}\sum_a (\bar{E}_a E_a + E_a\bar{E}_a + [\bar{E}_a,E_a])(s(X_1,\dots,X_q))\\
&\qquad - \sum_a\sum_{i=1}^{q}(-1)^{i-1}[X_i,\bar{E}_a](s(E_a,X_1,\dots,\widehat{X_i},\dots,X_q)).
\end{aligned}$$

We rewrite this formula by using the fact that $[\bar{E}_a,E_a]$ and $[X_i,\bar{E}_a]$ are elements of $\mathfrak{k}^{\mathbf{C}}$. Since a section $s\in C^\infty(E\otimes T^{0,q+1}M)\cong C^\infty(G,K;V\otimes\wedge^q\mathfrak{m}_-^*)$ satisfies

$$s(gk)(X_1,\dots,X_q) = \chi(k^{-1})s(g)(\mathrm{Ad}(k)X_1,\dots,\mathrm{Ad}(k)X_q))$$

for $g\in G$, $k\in K$, and $X_i\in\mathfrak{m}_-$ ($i=1,\dots,q$), we have for $Y\in\mathfrak{k}$ or $\mathfrak{k}^{\mathbf{C}}$

$$\begin{aligned}
(Ys)(X_1,\dots,X_q) &= -\chi(Y)s(X_1,\dots,X_q)\\
&\quad + \sum_{i=1}^{q}(-1)^{i-1}s([Y,X_i],X_1,\dots,\widehat{X_i},\dots,X_q).
\end{aligned}$$

Moreover, by the Jacobi identity and the fact $[\mathfrak{m}_-,\mathfrak{m}_-]=\{0\}$, we can easily show the following.

$$[[\bar{E}_a,E_a],X_i] = -[E_a,[\bar{E}_a,X_i]],\quad [[X_i,\bar{E}_a],X_j] = [[X_j,\bar{E}_a],X_i].$$

Therefore we have

$$\begin{aligned}
&(\Box s)(X_1,\dots,X_q)\\
&= -\frac{1}{2}\sum_a(\bar{E}_a E_a + E_a\bar{E}_a)(s(X_1,\dots,X_q))\\
&\quad + \frac{1}{2}\sum_a \chi([\bar{E}_a,E_a])s(X_1,\dots,X_q)\\
&\quad - \frac{1}{2}\sum_a\sum_{i=1}^{q}(-1)^{i-1}s([[\bar{E}_a,E_a],X_i],X_1,\dots,\widehat{X_i},\dots,X_q)
\end{aligned}$$

$$+\sum_a\sum_{i=1}^{q}(-1)^{i-1}\chi([X_i,\overline{E}_a])s(E_a,X_1,\ldots,\widehat{X_i},\ldots,X_q)$$
$$-\sum_a\sum_{i=1}^{q}(-1)^{i-1}s([[X_i,\overline{E}_a],E_a],X_1,\ldots,\widehat{X_i},\ldots,X_q)$$
$$+\sum_a\sum_{j<i}(-1)^{i+j}s([[X_i,\overline{E}_a],X_j],E_a,X_1,\ldots,\widehat{X_j},\ldots,\widehat{X_i},\ldots,X_q)$$
$$-\sum_a\sum_{i<j}(-1)^{i+j}s([[X_i,\overline{E}_a],X_j],E_a,X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q).$$
$$=-\frac{1}{2}\sum_a(\overline{E}_aE_a+E_a\overline{E}_a)(s(X_1,\ldots,X_q))$$
$$-\frac{1}{2}\sum_a\sum_{i=1}^{k}(-1)^{i-1}s([E_a,[\overline{E}_a,X_i]],X_1,\ldots,\widehat{X_i},\ldots,X_q)$$
$$+\frac{1}{2}\sum_a\chi([\overline{E}_a,E_a])s(X_1,\ldots,X_q)$$
$$-\sum_a\sum_{i=1}^{q}(-1)^{i-1}\chi([X_i,\overline{E}_a])s(E_a,X_1,\ldots,\widehat{X_i},\ldots,X_q).$$

We take an orthonormal basis $\{Y_b\}$ of $\mathfrak{k}$. The Casimir operator $C_{\mathfrak{g}}$ of $\mathfrak{g}$ is given by

$$C_{\mathfrak{g}}=\sum_a(\overline{E}_aE_a+E_a\overline{E}_a)+\sum_b Y_bY_b,$$

and the Casimir operator $C_{\mathfrak{k}}$ of $\mathfrak{k}$ is given by

$$C_{\mathfrak{k}}=\sum_b Y_bY_b.$$

We set

$$\chi(C_{\mathfrak{k}})=\sum_b\chi(Y_b)\chi(Y_b).$$

The action of $C_{\mathfrak{k}}$ on s is computed as follows:

$$\sum_b F_b(F_b(s(X_1,\ldots,X_q)))$$
$$=-\sum_b F_b(\chi(F_b)s(X_1,\ldots,X_q))$$
$$+\sum_b\sum_{i=1}^{q}(-1)^{i-1}F_b(s([F_bX_i],X_1,\ldots,\widehat{X_i},\ldots,X_q))$$
$$=-\sum_b\chi(F_b)F_b(s(X_1,\ldots,X_q))$$
$$+\sum_b\sum_{i=1}^{q}(-1)^{i-1}F_b(s([F_bX_i],X_1,\ldots,\widehat{X_i},\ldots,X_q))$$
$$=\sum_b\chi(F_b)\chi(F_b)s(X_1,\ldots,X_q)$$
$$-2\sum_b\sum_{i=1}^{q}(-1)^{i-1}\chi(F_b)s([F_b,X_i],X_1,\ldots,\widehat{X_i},\ldots,X_q)$$

$$+\sum_b\sum_{i=1}^{q}(-1)^{i-1}s([F_b,[F_b,X_i]],X_1,\ldots,\widehat{X_i},\ldots,X_q)$$
$$-\sum_b\sum_{j<i}(-1)^{i+j}s([F_b,X_j],[F_b,X_i],X_1,\ldots,\widehat{X_j},\ldots,\widehat{X_i},\ldots,X_q)$$
$$+\sum_b\sum_{i<j}(-1)^{i+j}s([F_b,X_j],[F_b,X_i],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q).$$

The last two terms are identical. By representing $[F_b,X_i]\in\mathfrak{m}_-$ with respect to the basis $\{E_a\}$, we have

$$\begin{aligned}
&\sum_b s([F_b,X_i],[F_b,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q)\\
&=\sum_b\sum_a s(B(\overline{E}_a,[F_b,X_i])E_a,[F_b,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q)\\
&=\sum_a\sum_b s(E_a,[B([X_i,\overline{E}_a],F_b)F_b,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q)\\
&=\sum_a s(E_a,[[X_i,\overline{E}_a],X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q)\\
&=\sum_a s(E_a,[[X_j,\overline{E}_a],X_i],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q)\\
&=\sum_b s([F_b,X_j],[F_b,X_i],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q)\\
&=0.
\end{aligned}$$

Thus they vanish.

For the third term, We have

$$\begin{aligned}
\sum_b[F_b,[F_b,X_i]]&=\sum_b\sum_a[F_b,B(\overline{E}_a,[F_b,X_i])E_a]\\
&=\sum_a\sum_b[B([X_i,\overline{E}_a],F_b)F_b,E_a]\\
&=\sum_a[[X_i,\overline{E}_a],E_a]\\
&=\sum_a[E_a,[\overline{E}_a,X_i]].
\end{aligned}$$

We also have

$$\begin{aligned}
&\sum_b\chi(F_b)s([F_b,X_i],X_1,\ldots,\widehat{X_i},\ldots,X_q)\\
&=\sum_b\sum_a\chi(F_b)s(B(\overline{E}_a,[F_b,X_i])E_a,X_1,\ldots,\widehat{X_i},\ldots,X_q)\\
&=\sum_a\sum_b\chi(B([X_i,\overline{E}_a],F_b)F_b)s(E_a,X_1,\ldots,\widehat{X_i},\ldots,X_q)\\
&=\sum_a\chi([X_i,\overline{E}_a])s(E_a,X_1,\ldots,\widehat{X_i},\ldots,X_q).
\end{aligned}$$

Substituting these formulas, we get

$$0=-\sum_b F_b(F_b(s(X_1,\dots,X_q))$$
$$+\sum_b \chi(F_b)\chi(F_b)s(X_1,\dots,X_q)$$
$$+\sum_a\sum_{i=1}^{q}(-1)^{i-1}s([E_a,[\overline{E}_a,X_i]],X_1,\dots,\widehat{X_i},\dots,X_q)$$
$$-2\sum_a\sum_{i=1}^{q}(-1)^{i-1}\chi([X_i,\overline{E}_a])s(E_a,X_1,\dots,\widehat{X_i},\dots,X_q).$$

Finally, our formula for $\bar{\partial}$-Laplacian $\Box$ becomes

$$(\Box s)(X_1,\dots,X_q)$$
$$=-\frac{1}{2}(\sum_a(\overline{E}_aE_a+E_a\overline{E}_a)+\sum_b F_bF_b)(s(X_1,\dots,X_q))$$
$$+\frac{1}{2}\sum_b\chi(F_b)\chi(F_b)s(X_1,\dots,X_q)$$
$$+\frac{1}{2}\chi(\sum_a[\overline{E}_a,E_a])s(X_1,\dots,X_q).$$

We denote by R the element $\sum_a[\overline{E}_a,E_a]\in\mathfrak{k}$ in the above formula, which is in the center of $\mathfrak{k}$ and, in fact, satisfies $\mathrm{Ad}(k)R=R$ $(k\in K)$.

Theorem 1. *The $\bar{\partial}$-Laplacian $\Box$ on $C^\infty(E\otimes T^{(0,q)}M)$ for a compact Hermitian symmetric space $M=G/K$ is given by*

$$(\Box s)(X_1,\dots,X_q)=-\frac{1}{2}C_{\mathfrak{g}}(s(X_1,\dots,X_q))+\frac{1}{2}\chi(C_{\mathfrak{k}}+R)s(X_1,\dots,X_q).$$

This formula enables us to compute the eigenvalues of $\bar{\partial}$-Laplacian $\Box$ representation theoretically.

For any associated vector bundle E over the homogeneous space G/K, the group G acts on $C^\infty(E)=C^\infty(G\times_K V)=C^\infty(G,K;V)$ as follows:

$$(g\cdot s)(g')=s(g^{-1}g'),\qquad s\in C^\infty(G,K;V),\quad g,g'\in G.$$

Since $C_{\mathfrak{g}}$ is an invariant differential operator, this action on $C^\infty(E\otimes T^{(0,q)}M)$ for a compact Hermitian symmetric space $M=G/K$ commutes with the $\bar{\partial}$-Laplacian $\Box$, and each eigenspace of $\Box$ is a G-invariant subspace. As the operator $\Box$ is elliptic, the eigenspace for each eigenvalue is finite-dimensional. Therefore we have the decomposition of $C^\infty(E\otimes T^{(0,q)}M)$ as the direct sum of irreducible (finite-dimensional) G-modules.

When the space of smooth sections $C^\infty(G,K;V)$ decomposes into the direct sum of irreducible G-modules, the summand is the image of a G-homomorphism from an irreducible G-module (ρ,U) to $C^\infty(G,K;V)$. The space of G-homomorphisms $\mathrm{Hom}_G(U,C^\infty(G,K;V))$ is identified with the space of K-homomorphisms $\mathrm{Hom}_K(U,V)$ by Frobenius' reciprocity law: The correspondence between $\Phi\in\mathrm{Hom}_G(U,C^\infty(G,K;V))$ and $\Psi\in\mathrm{Hom}_K(U,V)$ is given by

$$\Phi(u)(e)=\Psi(u),\qquad \Phi(u)(g)=\Psi(\rho(g^{-1})u),\qquad u\in U,\quad g\in G.$$

By Schur's Lemma, the space $\mathrm{Hom}_K(U, V)$ can be computed by the branching law, which describes how an irreducible G-module decomposes into the direct sum of irreducible K-modules when the G-action is restricted to the K-action.

Since the Casimir operator acts on an irreducible module as multiplication by a contant given by Freudenthal's formula, the irreducible G-submodule of $C^\infty(E \otimes T^{(0,q)}M)$ is included in an eigenspace of $\bar\partial$-Laplacian $\square$, the eigenvalue of which can be computed by the highest weight that determines the irreducible G-module. The dimension of the irreducible G-module can be computed by Weyl's formula. Thus we can compute the eigenvalues and the dimension of each eigenspace for $\bar\partial$-Laplacian $\square$.

The detail is given in the next section for the case $G = U(n+1)$ and $K = U(1) \times U(n)$.

3. Line bundles over complex projective spaces.

We shall compute the analytic torsion of a positive power of line bundle associated with hyperplane section over the complex projective spaces.

Let G be the unitary group $U(n+1)$ acting on the $(n+1)$-dimensional complex vector space $\mathbf{C}^{n+1}$ equipped with the standard Hermitian inner product. The group G acts transitively on the set of one-dimensional complex vector subspaces, that is, the complex projective space $P^n(\mathbf{C})$. We take the subspace containing ${}^t(1 \ \ 0 \ \ \cdots \ \ 0)$ as the origin o, and denote by K the isotropy subgroup at o. Thus we consider $P^n(\mathbf{C})$ as the homogeneous space $G/K = U(n+1)/\,U(1) \times U(n)$.

The Lie algebra $\mathfrak{g}$ of G is the space $\mathfrak{u}(n+1)$ of skew-Hermitian matrices of degree $n+1$. The invariant inner product B is given by $B(X, Y) = -\,\mathrm{trace}(XY)$, $(X, Y \in \mathfrak{u}(n+1))$. The diagonal matrix $j = \mathbf{diag}(1, \sqrt{-1}, \ldots, \sqrt{-1})$ gives a complex structure on the orthogonal complement $\mathfrak{m}$ in $\mathfrak{g}$ of the Lie subalgebra $\mathfrak{k}$ corresponding to the subgroup K. The anti-holomorphic part $\mathfrak{m}_-$ is spanned by $\{E_a\}_{a=1}^n$, the (i,j)-component $(E_a)_{ij}$ $(0 \le i,j \le n)$ of which is given by $(E_a)_{ij} = \delta_{i0}\delta_{ja}$. The basis $\{E_a\}$ is orthonormal. We set $\overline{E}_a = {}^tE_a$, and we get the orthonormal basis $\{\overline{E}_a\}$ of $\mathfrak{m}_+$.

The subgroup T of diagonal matrices in G is a maximal toral subgroup. It is also a maximal toral subgroup of K. The Lie algebra $\mathfrak{t}$ is the subalgebra of diagonal matrices with pure imaginary components. The elements $\{\lambda_i\}_{i=0}^n$ in $(\mathfrak{t}^{\mathbf{C}})^*$ defined by $\lambda_i(\mathbf{diag}(\mu_0, \mu_1, \ldots, \mu_n)) = \mu_i$ forms an orthonormal basis of $(\mathfrak{t}^{\mathbf{c}})^*$. We fix a lexicographical ordering in the real span of $\{\lambda_i\}$ such that $\lambda_0 > \lambda_1 > \cdots > \lambda_n$.

An irreducible G-module is designated by its highest weight $\Lambda_G = h_0\lambda_0 + h_1\lambda_1 + \cdots + h_n\lambda_n$, where $h_0, h_1, \ldots, h_n$ are integers satisfying $h_0 \ge h_1 \ge \cdots \ge h_n$. We denote it by $V_G(\Lambda_G)$. An irreducible K-module is designated by its highest weight $\Lambda_K = k_0\lambda_0 + k_1\lambda_1 + \cdots + k_n\lambda_n$, where $k_0, k_1, \ldots, k_n$ are integers satisfying $k_1 \ge \cdots \ge k_n$. We denote it by $V_K(\Lambda_K)$. The following branching law is well-known.

Proposition 2. *For an irreducible G-module $V_G(\Lambda_G)$ and an irreducible K-module $V_K(\Lambda_K)$, the dimension of* $\mathrm{Hom}_K(V_G(\Lambda_G), V_K(\Lambda_K))$ *is at most one. We have* $\dim \mathrm{Hom}_K(V_G(\Lambda_G), V_K(\Lambda_K)) = 1$ *if and only if the inequalities $h_0 \ge k_1 \ge h_1 \ge k_2 \ge h_2 \ge \cdots \ge k_n \ge h_n$ and the equality $k_0 = \sum_{i=0}^n h_i - \sum_{i=1}^n k_i$ are satisfied.*

We consider a one-dimensional K-module $(\chi_m, \mathbf{C})$ the action of which is defined by $\chi_m((e^{\sqrt{-1}\theta}, k'))z = e^{\sqrt{-1}m\theta}z$, $(k = (e^{\sqrt{-1}\theta}, k') \in K, z \in \mathbf{C})$. For $m = 1$, the associated line bundle $L_1 = G \times_K \mathbf{C}$ is the tautological line bundle of $P^n(\mathbf{C})$; for $m = -1$, the associated line bundle L_{-1} is its dual bundle, that is, the line bundle associated with the hyperplane divisor. We shall treat the line bundles L_{-m} for negative integers $-m$.

The vector bundle $L_{-m}\otimes T^{(0,q)}P^n(\mathbf{C})$ is associated with the K-module $\chi_{-m}\otimes\wedge^q\mathfrak{m}_-^*\cong\chi_{-m}\otimes\wedge^q\mathfrak{m}_+$, which is an irreducible K-module with the highest weight $-(m+q)\lambda_0+\Lambda_q$, where we set $\Lambda_q=\sum_{i=1}^q\lambda_i$.

Proposition 3. *The list of highest weights Λ_G for irreducible G modules $V_G(\Lambda_G)$ with* $\dim\mathrm{Hom}_K(V_G(\Lambda_G), V_K(-(m+q)\lambda_0+\Lambda_q))=1$ *is given as follows:*

$$\begin{aligned}
q=0:&\quad \Lambda_G=-m\lambda_n,\ (k+1)\lambda_0-(m+k+1)\lambda_n \quad (k\geq 0).\\
q=1:&\quad \Lambda_G=(k+1)\lambda_0-(m+k+1)\lambda_n \quad (k\geq 0),\\
&\quad\qquad (k+1)\lambda_0+\Lambda_1-(m+k+2)\lambda_n \quad (k\geq 0).\\
2\leq q\leq n-1:&\quad \Lambda_G=(k+1)\lambda_0+\Lambda_{q-1}-(m+k+q)\lambda_n \quad (k\geq 0),\\
&\quad\qquad (k+1)\lambda_0+\Lambda_q-(m+k+q+1)\lambda_n \quad (k\geq 0).\\
q=n:&\quad \Lambda_G=(k+1)\lambda_0+\Lambda_{n-1}-(m+k+n)\lambda_n \quad (k\geq 0).
\end{aligned}$$

By Freudenthal's formula, we know that the Casimir operator $C_{\mathfrak{g}}$ acts on an irreducible G-submodule of $C^\infty(L_{-m}\otimes T^{(0,q)}P^n(\mathbf{C}))$ isomorphic to $V_G(\Lambda_G)$ as an scalar multiplication by $B(\Lambda_G+2\delta_G,\Lambda_G)$, where δ_G is the half of the sum of all the positive roots of $\mathfrak{g}$. In the same way, the action of $\chi_{-m}(C_{\mathfrak{k}})$ can be computed, and is equal to $-m^2$. It is easy to show that the action of $\chi_{-m}(R)=\chi_{-m}(\sum_a[\bar{E}_a,E_a])$ is equal to $-mn$. Using Theorem 1, we can compute the eigenvalues of $\bar{\partial}$-Laplacian $\Box$ on $V_G(\Lambda_G)$. We notice that these values do not depend on q.

Proposition 4. *The action of $\bar{\partial}$-Laplacian $\Box$ on the irreducible G-submodule in $C^\infty(L_{-m}\otimes T^{(0,q)}P^n(\mathbf{C}))$ isomorphic to $V_G(\Lambda_G)$ is a scalar multiplication. The value is given as follows:*

Λ_G	$\Box$
$-m\lambda_n$	0
$(k+1)\lambda_0-(k+m+1)\lambda_n$	$(k+1)(k+m+n+1)$
$(k+1)\lambda_0+\Lambda_q-(k+m+q+1)\lambda_n$	$(k+q+1)(k+m+n+1)$

The dimension of an irreducible G-module $V_G(\Lambda_G)$ is computed by Weyl's formula:

$$\dim V_G(\Lambda_G)=\prod_{\alpha\in\Delta_+}\frac{B(\Lambda_G+\delta_G,\alpha)}{B(\delta_G,\alpha)},$$

where Δ_+ denotes the set of positive roots of $\mathfrak{g}$.

Proposition 5. *An irreducible G-submodule in $C^\infty(L_{-m}\otimes T^{(0,q)}P^n(\mathbf{C}))$ isomorphic to $V_G(\Lambda_G)$ has the following dimension:*

$$\dim V_G(-m\lambda_n)=\binom{m+n}{n},$$

$$\dim V_G((k+1)\lambda_0-(k+m+1)\lambda_n)$$
$$=\binom{k+n}{n-1}\binom{k+m+n+1}{n}\frac{2k+m+n+2}{k+m+n+1},$$
$$\dim V_G((k+1)\lambda_0+\Lambda_q-(k+m+q+1)\lambda_n)$$
$$=\binom{k+q}{q}\binom{k+n}{n-q-1}\binom{k+m+n+q+1}{n}\frac{2k+m+n+q+2}{k+m+n+1}.$$

4. THE ANALYTIC TORSION OF L_{-m}.

Let E be a holomorphic vector bundle with a Hermitian metric over an n-dimensional Kähler manifold M. By using the spectrum $\mathrm{Spec}(\Box_q)$ of $\bar\partial$-Laplacian $\Box_q$ on $C^\infty(E\otimes T^{(0,q)}M)$, that is, all the eigenvalues of $\Box_q$, we define a spectral zeta function $\zeta_{E,q}$:

$$\zeta_{E,q}(s)=\sum_{\lambda\in\mathrm{Spec}(\Box_q),\,\lambda>0}\mathrm{mult}(\lambda)\lambda^{-s},$$

where $\mathrm{mult}(\lambda)$ is the dimension of the eigenspace associated with the eigenvalue λ. It is known that the spectral zeta function is defined on a half complex plane for $\Re s$ large enough, has a meromorphic continuation to the whole complex plane, and is analytic at $s=0$. We define the spectral zeta function ζ_E of the Dolbeault complex for E by

$$\zeta_E(s)=\sum_{q=0}^{n}(-1)^q q\zeta_{E,q}(s).$$

The analytic torsion A-Tor(E) of E is the value of the differential of ζ_E at $s=0$:

$$\text{A-Tor}(E)=\zeta'_E(0).$$

In view of Propositions 3, 4, and 5, the spectral zeta functions $\zeta_{L_{-m},q}$ and $\zeta_{L_{-m}}$ for the line bundle L_{-m} with negative $-m$ over the complex projective space $P^n(\mathbf{C})$ are given as follows:

$$\zeta_{L_{-m},0}(s)=\sum_{k=0}^{\infty}\binom{k+n}{n-1}\binom{k+m+n+1}{n}\frac{2k+m+n+2}{k+m+n+1}\times\{(k+1)(k+m+n+1)\}^{-s},$$

$$\zeta_{L_{-m},q}(s)=\sum_{k=0}^{\infty}\binom{k+q-1}{q-1}\binom{k+n}{n-q}\binom{k+m+n+q}{n}\frac{2k+m+n+q+1}{k+m+n+1}\times\{(k+q)(k+m+n+1)\}^{-s}$$
$$+\sum_{k=0}^{\infty}\binom{k+q}{q}\binom{k+n}{n-q-1}\binom{k+m+n+q+1}{m}\frac{2k+m+n+q+2}{k+m+n+1}\times\{(k+q+1)(k+m+n+1)\}^{-s}\quad(1\le q\le n-1),$$

$$\zeta_{L_{-m},n}(s)=\sum_{k=0}^{\infty}\binom{k+n-1}{n}\binom{k+m+2n}{n}\frac{2k+m+2n+1}{k+m+n+1}$$
$$\times\{(k+n)(k+m+n+1)\}^{-s},$$

$$\zeta_{L_{-m}}(s)=\sum_{q=1}^{n}(-1)^q\sum_{k=0}^{\infty}\binom{k+q-1}{q-1}\binom{k+n}{n-q}\binom{k+m+n+q}{n}$$
$$\times\frac{2k+m+n+q+1}{k+m+n+1}\{(k+q)(k+m+n+1)\}^{-s}$$
$$=n\sum_{q=1}^{n}(-1)^q\binom{n-1}{q-1}\sum_{k=0}^{\infty}\binom{k+n}{n}\binom{k+m+n+q}{n}$$
$$\times(2k+m+n+q+1)\{(k+q)(k+m+n+1)\}^{-s-1}.$$

For positive integers n, m, and q, we set

$$\zeta_{n,m,q}(s)=\sum_{k=0}^{\infty}\binom{k+n}{n}\binom{k+m+n+q}{m}(2k+m+n+q+1)$$
$$\times\{(k+q)(k+m+n+1)\}^{-s-1}.$$

We call attention to the fact that the polynomial

$$\binom{k+n}{n}\binom{k+m+n+q}{n}(2k+m+n+q+1)$$

in k is divisible by $(k+q)(k+m+n+1)$ for $1\le q\le n$.

We consider in general a Dirichlet sum of the form

$$\zeta_{P,\delta_1,\delta_2}(s)=\sum_{k=0}^{\infty}P(k)\{(k+\delta_1)(k+\delta_2)\}^{-s},$$

where δ_1 and δ_2 are positive integers with $\delta_1<\delta_2$, and $P(k)$ is a polynomial in k of degree $2n+1$ satisfying

$$P\left(-x-\frac{\delta_1+\delta_2}{2}\right)=-P\left(x-\frac{\delta_1+\delta_2}{2}\right).$$

In other words, $P(k)$ is a polynomial of the form

$$P(k)=\sum_{d=0}^{n}c_d\left(k+\frac{\delta_1+\delta_2}{2}\right)^{2d+1},\qquad c_d=\frac{1}{(2d+1)!}P^{(2d+1)}\left(-\frac{\delta_1+\delta_2}{2}\right).$$

We shall give an expansion of $\zeta_{P,\delta_1,\delta_2}(s)$ in terms of Hurwitz zeta functions $\zeta(s,a)$:

$$\zeta(s,a)=\sum_{k=0}^{\infty}(k+a)^{-s}.$$

Theorem 6. *For positive integers δ_1, δ_2 with $\delta_1<\delta_2$ and a polynomial P as above, we have*

$$\zeta_{P,\delta_1,\delta_2}(s)=\sum_{d=0}^{n}c_d\sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}\left(\frac{\delta_2-\delta_1}{2}\right)^{2m}\zeta\left(2s+2m-2d-1,\frac{\delta_1+\delta_2}{2}\right).$$

In the following, we set

$$\delta_+ = \frac{\delta_1 + \delta_2}{2}, \qquad \delta_- = \frac{\delta_2 - \delta_1}{2}.$$

We begin the proof by the following simple lemma.

Lemma 7. *For any non-negative integer k and any complex number s except for non-positive integers, we have*

$$\{(k+\delta_1)(k+\delta_2)\}^{-s} = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{m!\Gamma(s)} \delta_-^{2m} (k+\delta_+)^{-2s-2m}.$$

Proof. We apply a binomial expansion

$$(1-x)^{-s} = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{m!\Gamma(s)} x^m,$$

which holds for any complex number s except for non-positive integers and for any real number x with $|x| < 1$, to the last part of the following equation.

$$\begin{aligned}\{(k+\delta_1)(k+\delta_2)\}^{-s} &= \{(k+\delta_+)^2 - \delta_-^2\}^{-s} \\ &= (k+\delta_+)^{-2s}\left(1 - \left(\frac{\delta_-}{k+\delta_+}\right)^2\right)^{-s}.\end{aligned}$$

□

Proof of Theorem 6. It is enough to consider the case P is a monomial $(k+\delta_+)^{2d+1}$. For $\Re s = s_0 > d+1$, by the inequality

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left|\frac{\Gamma(s+m)}{m!\Gamma(s)}\delta_-^{2m}(k+\delta_+)^{2d+1-2s-2m}\right| \\ &\leq \sum_{m=0}^{\infty}\left(\frac{\Gamma(|s|+m)}{m!\Gamma(|s|)}\delta_-^{2m}\sum_{k=0}^{\infty}(k+\delta_+)^{2d+1-2s_0-2m}\right) \\ &\leq \sum_{m=0}^{\infty}\left(\frac{\Gamma(|s|+m)}{m!\Gamma(|s|)}\delta_-^{2m}\cdot C_m \delta_+^{2(d+1-s_0-m)}\right) \\ &\leq C_0 \delta_+^{2(d+1-s_0)}\left(1 - \left(\frac{\delta_-}{\delta_+}\right)^2\right)^{-|s|},\end{aligned}$$

$$\left(C_m = \frac{1}{2(s_0 - d - 1 + m)} + \frac{1}{\delta_+}\right)$$

we may change the order of the summations, and thus we have

$$\begin{aligned}&\sum_{k=0}^{\infty}(k+\delta_+)^{2d+1}\{(k+\delta_1)(k+\delta_2)\}^{-s} \\ &= \sum_{m=0}^{\infty}\left(\frac{\Gamma(s+m)}{m!\Gamma(s)}\delta_-^{2m}\sum_{k=0}^{\infty}(k+\delta_+)^{2d+1-2s-2m}\right) \\ &= \sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}\delta_-^{2m}\zeta(2s+2m-2d-1, \delta_+).\end{aligned}$$

We need to clarify the meaning of the equality. For any positive integer N, we have

$$\sum_{k=0}^{\infty} (k+\delta_+)^{2d+1}\{(k+\delta_1)(k+\delta_2)\}^{-s}$$
$$= \sum_{m=0}^{N-1} \frac{\Gamma(s+m)}{m!\Gamma(s)}\delta_-^{2m}\zeta(2s+2m-2d-1,\delta_+)$$
$$+ \frac{\Gamma(s+N)}{\Gamma(s)}\delta_-^{2N} Z_N(s),$$

where $Z_N(s)$ is given by

$$Z_N(s) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{\Gamma(s+N+\ell)}{(N+\ell)!\Gamma(s+N)}\delta_-^{2\ell}(k+\delta_+)^{2d+1-2s-2N-2\ell}.$$

We put $S = d+2-N$. In a closed domain $D_{R,S} = \{ s \in \mathbf{C} \mid |s| \le R, \Re s \ge S \}$, we have

$$\left| \frac{d}{ds} \frac{\Gamma(s+N+\ell)}{(N+\ell)!\Gamma(s+N)}\delta_-^{2\ell}(k+\delta_+)^{2d+1-2s-2N-2\ell} \right|$$
$$\le \frac{\ell\Gamma(R+N+\ell)}{(N+\ell)!\Gamma(R+N+1)}\delta_-^{2\ell}(k+\delta_+)^{-3-2\ell}$$
$$+ \frac{\Gamma(R+N+\ell)}{(N+\ell)!\Gamma(R+N)}\delta_-^{2\ell}(k+\delta_+)^{-3-2\ell}\log(k+\delta_+)$$
$$\le \frac{\ell\Gamma(R+N+\ell)}{\ell!\Gamma(R+N+1)}\delta_-^{2\ell}(k+\delta_+)^{-3-2\ell}$$
$$+ \frac{\Gamma(R+N+\ell)}{\ell!\Gamma(R+N)}\delta_-^{2\ell}\frac{1}{e}(k+\delta_+)^{-2-2\ell},$$

where we used the fact that $e\log x \le x$ for $x > 0$. Since the last formula of the above inequality is summable over k and ℓ, $Z_N(s)$ is holomorphic on $\Re s > d+1-N$.

In short, the Dirichlet sum $\zeta_{P,\delta_1,\delta_2}(s)$ has the analytic continuation, which is a meromorphic function on $\mathbf{C}$ given by the infinite sum of Hurwitz zeta functions as in Theorem 6. □

We notice that this theorem is essentially obtained in [3], but that our proof is much simpler, and that our formulation is indispensable for our computation of A-Tor(L_{-m}).

The analytic continuation of Hurwitz zeta function is well-known. It is meromorphic on $\mathbf{C}$ and has a single simple pole at 1. The Laurent expansion at 1 is given by

$$\zeta(s,a) = \frac{1}{s-1} - \Psi(a) + a_1(s-1) + a_2(s-1)^2 + \cdots,$$

where $\Psi(a)$ is the digamma function:

$$\Psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}.$$

To compute the analytic torsion of L_{-m}, we shall calculate as follows:

$$\frac{d}{ds}\left(\sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}\delta_-^{2m}\zeta(2s+2m-2d-1,\delta_+)\right)\Bigg|_{s=0}$$

$$=2\zeta'(-2d-1,\delta_+)$$

$$+\sum_{m=1}^{d}\frac{1}{m}\delta_-^{2m}\zeta(2m-2d-1,\delta_+)$$

$$+\frac{d}{ds}\left(\frac{s(s+1)\cdots(s+d)}{(d+1)!}\delta_-^{2d+2}\left(\frac{1}{2s}-\Psi(\delta_+)\right)\right)\Bigg|_{s=0}$$

$$+\sum_{m=d+2}^{\infty}\frac{1}{m}\delta_-^{2m}\zeta(2m-2d-1,\delta_+).$$

The third term of the right hand side is equal to

$$\frac{\delta_-^{2d+2}}{d+1}\left(\frac{H(d)}{2}-\Psi(\delta_+)\right),$$

where $H(d)$ is the harmonic number:

$$H(d)=\sum_{n=1}^{d}\frac{1}{n}=1+\frac{1}{2}+\cdots+\frac{1}{d}.$$

We set $H(0)=0$.

To compute the fourth term, we first rewrite it by the integral expression of the Hurwitz zeta function.

$$\begin{aligned}\zeta(s,a)&=\sum_{k=0}^{\infty}(k+a)^{-s}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(s)}\int_0^{\infty}u^{s-1}e^{-(k+a)u}du\\&=\frac{1}{\Gamma(s)}\int_0^{\infty}u^{s-1}\frac{e^{-au}}{1-e^{-u}}du,\end{aligned}$$

which holds for $\Re s>1$.

$$\begin{aligned}&\sum_{m=d+2}^{\infty}\frac{1}{m}\delta_-^{2m}\zeta(2m-2d-1,\delta_+)\\&=\sum_{\ell=0}^{\infty}\frac{1}{\ell+d+2}\delta_-^{2\ell+2d+4}\zeta(2\ell+3,\delta_+)\\&=\sum_{\ell=0}^{\infty}\frac{1}{(\ell+d+2)\cdot(2\ell+2)!}\delta_-^{2\ell+2d+4}\int_0^{\infty}u^{2\ell+2}\frac{e^{-\delta_+u}}{1-e^{-u}}du.\end{aligned}$$

We consider a holomorphic function $F(s)$ defined by

$$F(s)=\sum_{\ell=0}^{\infty}\frac{1}{(\ell+d+2)\cdot(2\ell+2)!}\delta_-^{2\ell+2d+4}\int_0^{\infty}u^{s+2\ell+2}\frac{e^{-\delta_+u}}{1-e^{-u}}du$$

for $\Re s>-2$. Notice that

$$\sum_{\ell=0}^{\infty}\frac{1}{(\ell+d+2)\cdot(2\ell+2)!}\delta_-^{2\ell+2d+4}u^{2\ell}\leq\frac{\delta_-^{2d+4}}{d+2}e^{\delta_-u}.$$

For $\Re s$ large enough, using the formula

$$\frac{1}{(\ell+d+2)\cdot(2\ell+2)!}=2\sum_{p=0}^{2d+1}(-1)^p\frac{(2d+1)!}{(2d-p+1)!}\frac{1}{(2\ell+p+3)!},$$

we can rewrite $F(s)$ as follows:

$$\begin{aligned}
F(s)&=2\sum_{p=0}^{2d+1}(-1)^p\frac{(2d+1)!}{(2d-p+1)!}\sum_{\ell=0}^{\infty}\frac{\delta_-^{2\ell+2d+4}}{(2\ell+p+3)!}\int_0^\infty u^{s+2\ell+2}\frac{e^{-\delta_+u}}{1-e^{-u}}du\\
&=\sum_{p=0}^{2d+1}(-1)^p\frac{(2d+1)!\,\delta_-^{2d-p+1}}{(2d-p+1)!}\\
&\qquad\times\sum_{\ell=p+3}^{\infty}\frac{(1-(-1)^{\ell+p})\delta_-^\ell}{\ell!}\int_0^\infty u^{s-p-1+\ell}\frac{e^{-\delta_+u}}{1-e^{-u}}du\\
&=\sum_{p=0}^{2d+1}(-1)^p\frac{(2d+1)!\,\delta_-^{2d-p+1}}{(2d-p+1)!}\\
&\qquad\times\Bigg(\int_0^\infty u^{s-p-1}\frac{e^{-(\delta_+-\delta_-)u}-(-1)^pe^{-(\delta_++\delta_-)u}}{1-e^{-u}}du\\
&\qquad\quad-\sum_{\ell=0}^{p+2}\frac{(1-(-1)^{\ell+p})\delta_-^\ell}{\ell!}\int_0^\infty u^{s-p-1+\ell}\frac{e^{-\delta_+u}}{1-e^{-u}}du\Bigg)\\
&=\sum_{p=0}^{2d+1}(-1)^p\frac{(2d+1)!\,\delta_-^{2d-p+1}}{(2d-p+1)!}\\
&\qquad\times\Bigg(\Gamma(s-p)(\zeta(s-p,\delta_1)-(-1)^p\zeta(s-p,\delta_2))\\
&\qquad\quad-\sum_{\ell=0}^{p+1}\frac{(1-(-1)^{\ell+p})\delta_-^\ell}{\ell!}\Gamma(s-p+\ell)\zeta(s-p+\ell,\delta_+)\Bigg)\\
&=\sum_{p=0}^{2d+1}(-1)^p\frac{(2d+1)!\,\delta_-^{2d-p+1}}{(2d-p+1)!}\Gamma(s-p)(\zeta(s-p,\delta_1)-(-1)^p\zeta(s-p,\delta_2))\\
&\quad-\sum_{q=1}^{2d+1}\frac{(2d+1)!}{(2d-q+1)!}\delta_-^{2d-q+1}\Gamma(s-q)\zeta(s-q,\delta_+)\\
&\qquad\times\sum_{p=q}^{2d+1}((-1)^p-(-1)^{p+q})\binom{2d-q+1}{p-q}\\
&\quad-\sum_{p=0}^{2d+1}(-1)^p\frac{2(2d+1)!}{(2d-p+1)!(p+1)!}\delta_-^{2d+2}\Gamma(s+1)\zeta(s+1,\delta_+)\\
&=\sum_{p=0}^{2d+1}(-1)^p\frac{(2d+1)!\,\delta_-^{2d-p+1}}{(2d-p+1)!}\Gamma(s-p)(\zeta(s-p,\delta_1)-(-1)^p\zeta(s-p,\delta_2))\\
&\quad+2(2d+1)!\,\Gamma(s-2d-1)\zeta(s-2d-1,\delta_+)\\
&\quad-\frac{1}{d+1}\delta_-^{2d+2}\Gamma(s+1)\zeta(s+1,\delta_+).
\end{aligned}$$

We know the analytic continuations of the members of the last expression. For example, the Laurent expansion at $s=0$ of $\Gamma(s-p)\zeta(s-p,\delta_1)$ is given by

$$\begin{aligned}\Gamma(s-p)\zeta(s-p,\delta_1)&\\ &=(-1)^p\frac{\zeta(-p,\delta_1)}{p!}\frac{1}{s}+\frac{d}{ds}\left(\frac{\Gamma(s+1)\zeta(s-p,\delta_1)}{(s-1)(s-2)\cdots(s-p)}\right)\Bigg|_{s=0}+\cdots\\ &=(-1)^p\frac{\zeta(-p,\delta_1)}{p!}\frac{1}{s}+(-1)^p\frac{\zeta'(-p,\delta_1)+(\Psi(1)+H(p))\zeta(-p,\delta_1)}{p!}+\cdots.\end{aligned}$$

Since $F(s)$ is holomorphic at $s=0$, the poles should cancel out, which gives

$$\begin{aligned}0=\sum_{p=0}^{2d+1}\binom{2d+1}{p}\delta_-^{2d-p+1}(\zeta(-p,\delta_1)-(-1)^p\zeta(-p,\delta_2))&\\ -2\zeta(-2d-1,\delta_+)-\frac{\delta_-^{2d+2}}{d+1}&.\end{aligned}$$

The value $F(0)$ is given by the summation of the constant terms in the Laurent expansions:

$$\begin{aligned}F(0)=&\sum_{p=0}^{2d+1}\binom{2d+1}{p}\delta_-^{2d-p+1}((\zeta'(-p,\delta_1)+(\Psi(1)+H(p))\zeta(-p,\delta_1))\\ &\qquad -(-1)^p(\zeta'(-p,\delta_2)+(\Psi(1)+H(p))\zeta(-p,\delta_2)))\\ &-2(\zeta'(-2d-1,\delta_+)+(\Psi(1)+H(2d+1))\zeta(-2d-1,\delta))\\ &-\frac{\delta_-^{2d+2}}{d+1}(\Psi(1)-\Psi(\delta_+))\\ =&\sum_{p=0}^{2d+1}\binom{2d+1}{p}\delta_-^{2d-p+1}((\zeta'(-p,\delta_1)+H(p)\zeta(-p,\delta_1))\\ &\qquad -(-1)^p(\zeta'(-p,\delta_2)+H(p)\zeta(-p,\delta_2)))\\ &-2(\zeta'(-2d-1,\delta_+)+H(2d+1)\zeta(-2d-1,\delta))\\ &+\frac{\delta_-^{2d+2}}{d+1}\Psi(\delta_+).\end{aligned}$$

By substituting these formulas, we get the following:

Theorem 8. *For positive integers δ_1, δ_2 with $\delta_1<\delta_2$ and a polynomial P as in Theorem 6, we have*

$$\begin{aligned}\zeta'_{P,\delta_1,\delta_2}(0)=\sum_{d=0}^{n}c_d\Bigg(&\sum_{p=1}^{d}\frac{\delta_-^{2p}}{p}\zeta(-2(d-p)-1,\delta_+)+\frac{\delta_-^{2d+2}}{2d+2}H(d)\\ &+\sum_{p=0}^{2d+1}\binom{2d+1}{p}\delta_-^{2d-p+1}(\zeta'(-p,\delta_1)-(-1)^p\zeta'(-p,\delta_2))\\ &+\sum_{p=1}^{2d+1}\binom{2d+1}{p}\delta_-^{2d-p+1}H(p)(\zeta(-p,\delta_1)-(-1)^p\zeta(-p,\delta_2))\\ &-2H(2d+1)\zeta(-2d-1,\delta_+)\Bigg).\end{aligned}$$

Therefore $\zeta'_{P,\delta_1,\delta_2}(0)$ can be expressed by a finite sum of the special values of the well-known functions. We notice that $\zeta(-p,a)$ can be expressed as

$$\zeta(-p,a)=-\frac{1}{p+1}B_{p+1}(a),$$

using the Bernoulli polynomial B_{p+1} defined by

$$\sum_{k=0}^{\infty}\frac{B_k(a)x^k}{k!}=\frac{xe^{ax}}{e^x-1},$$

and that $\zeta'(-p,a)$ for a natural number a can be expressed as

$$\zeta'(-p,a)=\zeta'(-p)+\sum_{k=1}^{a-1}k^p\log k,$$

using the Riemann zeta function $\zeta(s)=\zeta(s,1)$. We know that

$$B_1(a)=a-\frac{1}{2},\qquad B_2(a)=a^2-a+\frac{1}{6},\qquad B_3(a)=a^3-\frac{3}{2}a^2+\frac{1}{2}a.$$

These enable us to compute as follows. For $P(k)=k+\delta_+$, we have

$$\begin{aligned}\zeta'_{P,\delta_1,\delta_2}(0)&=\sum_{p=0}^{1}\delta_-^{1-p}(\zeta(-p,\delta_1)-(-1)^p\zeta(-p,\delta_2))-\delta_-^2\\&=2\zeta'(-1)+\left(\sum_{k=2}^{\delta_1-1}k\log k+\sum_{k=2}^{\delta_2-1}k\log k\right)-\delta_-\sum_{k=\delta_1}^{\delta_2-1}\log k-\delta_-^2,\end{aligned}$$

and, for $P(k)=(k+\delta_+)^3$, we have

$$\begin{aligned}\zeta'_{P,\delta_1,\delta_2}(0)&=\sum_{p=0}^{3}\delta_-^{3-p}\binom{3}{p}(\zeta(-p,\delta_1)-(-1)^p\zeta(-p,\delta_2))-\frac{2}{3}\delta_-^4\\&=2\zeta'(-3)+6\delta_-^2\zeta'(-1)\\&\quad+\left(\sum_{k=2}^{\delta_1-1}k^3\log k+\sum_{k=2}^{\delta_2-1}k^3\log k\right)-3\delta_-\sum_{k=\delta_1}^{\delta_2-1}k^2\log k\\&\quad+3\delta_-^2\left(\sum_{k=2}^{\delta_1-1}k\log k+\sum_{k=2}^{\delta_2-1}k\log k\right)-\delta_-^3\sum_{k=\delta_1}^{\delta_2-1}\log k\\&\quad-\frac{2}{3}\delta_-^4.\end{aligned}$$

By Theorem 8, We can directly compute the analytic torsion of the line bundle L_{-m} with negative $-m$ over the complex projective space $P^n(\mathbf{C})$. We shall give the detail for low-dimensional $P^n(\mathbf{C})$ in the next section.

5. The examples in low dimensions.

For the case $n=1$, $\zeta_{L_{-m}}(s)$ is given by

$$\zeta_{L_{-m}}(s) = \zeta_{1,m,1}(s) = \sum_{k=0}^{\infty}(2k+m+3)\{(k+1)(k+m+2)\}^{-s}.$$

We set $\delta_1 = 1$, $\delta_2 = m+2$, and $P(k) = 2(k+\delta_+)$. The analytic torsion A-Tor$(L_{-m}) \fallingdotseq \zeta'_{P,\delta_1,\delta_2}(0)$ is given by

$$\text{A-Tor}(L_{-m}) = 2\left(2\zeta'(-1) + \left(\sum_{k=2}^{\delta_1 - 1} k\log k + \sum_{k=2}^{\delta_2 - 1} k\log k\right) - \delta_- \sum_{k=\delta_1}^{\delta_2 - 1}\log k - \delta_-^2,\right)$$

$$= 4\zeta'(-1) + 2\sum_{k=2}^{m+1} k\log k - (m+1)\log((m+1)!) - \frac{(m+1)^2}{2}.$$

This result seems to be well-known.

For the case $n = 2$, $\zeta_{L_{-m}}(s)$ is given by

$$\zeta_{L_{-m}}(s) = 2(-\zeta_{2,m,1} + \zeta_{2,m,2}),$$

$$2\zeta_{2,m,1} = \sum_{k=0}^{\infty}(k+2)(k+m+2)\left(k+\frac{m+4}{2}\right)\{(k+1)(k+m+3)\}^{-s},$$

$$2\zeta_{2,m,2} = \sum_{k=0}^{\infty}(k+1)(k+m+4)\left(k+\frac{m+5}{2}\right)\{(k+2)(k+m+3)\}^{-s}.$$

For $2\zeta_{2,m,1}$, we set $\delta_1 = 1$, $\delta_2 = m+3$, and

$$P_1(k) = \left(k+\delta_+ - \frac{m}{2}\right)\left(k+\delta_+ + \frac{m}{2}\right)(k+\delta_+)$$

$$= (k+\delta_+)^3 - \left(\frac{m}{2}\right)^2(k+\delta_+),$$

$$\delta_+ = \frac{m+4}{2}, \qquad \delta_- = \frac{m+2}{2}.$$

For $2\zeta_{2,m,2}$, we set $\delta_1 = 2$, $\delta_2 = m+3$, and

$$P_2(k) = \left(k+\delta_+ - \frac{m+3}{2}\right)\left(k+\delta_+ + \frac{m+3}{2}\right)(k+\delta_+)$$

$$= (k+\delta_+)^3 - \left(\frac{m+3}{2}\right)^2(k+\delta_+),$$

$$\delta_+ = \frac{m+5}{2}, \qquad \delta_- = \frac{m+1}{2}.$$

We have

$$
\begin{aligned}
2\zeta'_{2,m,1}(0) &= \zeta'_{P_1,1,m+3}(0) \\
&= \Big(2\zeta'(-3) + 6\delta_-^2\zeta'(-1) + \sum_{k=2}^{m+2} k^3\log k - 3\delta_- \sum_{k=2}^{m+2} k^2 \log k \\
&\qquad + 3\delta_-^2 \sum_{k=2}^{m+2} k\log k - \delta_-^3 \sum_{k=2}^{m+2} \log k - \frac{2}{3}\delta_-^4\Big) \\
&\qquad - \left(\frac{m}{2}\right)^2 \left(2\zeta'(-1) + \sum_{k=2}^{m+2} k\log k - \delta_- \sum_{k=2}^{m+2} \log k - \delta_-^2\right) \\
&= 2\zeta'(-3) + \left(3\left(\frac{m+2}{2}\right)^2 - \left(\frac{m}{2}\right)^2\right)\left(2\zeta'(-1) + \sum_{k=2}^{m+2} k\log k\right) \\
&\qquad - \left(\frac{m+2}{2}\right)\left(\left(\frac{m+2}{2}\right)^2 - \left(\frac{m}{2}\right)^2\right)\sum_{k=2}^{m+2}\log k \\
&\qquad - 3\left(\frac{m+2}{2}\right)\sum_{k=2}^{m+2} k^2\log k + \sum_{k=2}^{m+2} k^3 \log k \\
&\qquad - \left(\frac{m+2}{2}\right)^2\left(\frac{2}{3}\left(\frac{m+2}{2}\right)^2 - \left(\frac{m}{2}\right)^2\right).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
2\zeta'_{2,m,2}(0) &= \zeta'_{P_2,2,m+3}(0) \\
&= 2\zeta'(-3) + \left(3\left(\frac{m+1}{2}\right)^2 - \left(\frac{m+3}{2}\right)^2\right)\left(2\zeta'(-1) + \sum_{k=2}^{m+2} k\log k\right) \\
&\qquad - \left(\frac{m+1}{2}\right)\left(\left(\frac{m+1}{2}\right)^2 - \left(\frac{m+3}{2}\right)^2\right)\sum_{k=2}^{m+2}\log k \\
&\qquad - 3\left(\frac{m+1}{2}\right)\sum_{k=2}^{m+2} k^2\log k + \sum_{k=2}^{m+2} k^3 \log k \\
&\qquad - \left(\frac{m+1}{2}\right)^2\left(\frac{2}{3}\left(\frac{m+1}{2}\right)^2 - \left(\frac{m+3}{2}\right)^2\right).
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\text{A-Tor}(L_{-m}) &= -2\zeta'_{2,m,1}(0) + 2\zeta'_{2,m,2}(0) \\
&= -(6m+9)\zeta'(-1) \\
&\qquad + \sum_{k=2}^{m+2}\left((m+1)(m+2) - \frac{3}{2}(2m+3) + \frac{3}{2}k^2\right)\log k \\
&\qquad + \frac{5}{12}m^3 + \frac{15}{3}m^2 + \frac{8}{3}m + \frac{19}{16}.
\end{aligned}
$$

We notice that, since we have

$$
\sum_{a+b+c=m} \log\left(\frac{a!b!c!}{(m+2)!}\right) = \sum_{k=2}^{m+2}\left((m+1)(m+2) - \frac{3}{2}(2m+3) + \frac{3}{2}k^2\right)\log k,
$$

where a, b, and c are non-negative integers, this result coincides with the result of [2], obtained by the arithmetic Riemann-Roch formula.

In higher dimensional cases, we will need some combinatorial relations for the coincidence, which may attract distinctive interest.

References

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